A PROOF OF THEOREM 14.2.1 (BASS' THEOREM) OF "SEPARABLE ALGEBRAS" BY T. J. FORD GRADUATE STUDIES IN MATHEMATICS, VOL. 183, 2017 July 18, 2018

TIMOTHY J. FORD

The purpose of this note is to prove the following theorem of Bass, which is stated without proof in [2, Theorem 14.2.1]. The proof given in [1, Proposition (4.6), p. 476] is K-theoretic. The proof given below is based on the method suggested in the paragraph immediately preceding [3, Theorem III.17] and utilizes only theorems proven in [2]. The main idea for the proof is the following lemma.

Lemma 0.1. *Let R be a ring and M a left R-module. For any $n > 0$, the assignment*

$$\operatorname{Hom}_R(M, M) \xrightarrow{\Delta} \operatorname{Hom}_R(M^{(n)}, M^{(n)})$$

that maps a homomorphism φ in $\operatorname{Hom}_R(M, M)$ to the corresponding diagonal homomorphism $\Delta(\varphi) = \oplus_{i=1}^{n} \varphi$ in $\operatorname{Hom}_R(M^{(n)}, M^{(n)})$ defines a monomorphism of rings. If R is commutative, Δ is an R-algebra homomorphism.

Proof. The proof is left to the reader. □

Theorem 0.2. *(H. Bass) Let R be a commutative ring and M an R-module. Then M is an R-progenerator if and only if there exists an R-module P such that $P \otimes_R M \cong R^{(s)}$ for some $s > 0$.*

Proof. If there exists an R-module P such that $P \otimes_R M \cong R^{(s)}$, then by [2, Proposition 1.3.4], both M and P are R-progenerators.

Assume M is an R-progenerator. First we show how to reduce to the case where M has constant rank. Assume M does not have constant rank. Let $e_1, \dots, e_t$ be the structure idempotents of M in R ([2, Corollary 2.3.6]). Write R_i for Re_i and M_i for Me_i. Then $R = R_1 \oplus \cdots \oplus R_t$, $M = M_1 \oplus \cdots \oplus M_t$, and M_i is an R_i-progenerator of constant rank. Assume there exists an R_i-module P_i such that $M_i \otimes_{R_i} P_i \cong R_i^{(s_i)}$ for some $s_i > 0$. Let s be the least common multiple of $\{s_1, \dots, s_t\}$. Then $M \otimes_R \left(P_1^{(s/s_1)} \oplus \cdots \oplus P_t^{(s/s_t)}\right) \cong R^{(s)}$.

Assume from now on that M has constant rank r. If M is free, then there is nothing to prove. Assume N is an R-progenerator such that $M \oplus N$ is free of rank rn and $n \geq 2$. Let S be a commutative faithfully flat R-algebra such that $M \otimes_R S$ and $N \otimes_R S$ are isomorphic to the free S-modules $S^{(r)}$ and $S^{(rn-r)}$, respectively. Then $(M \oplus N) \otimes_R S$ can be written as a direct sum $\oplus_{i=1}^{n} S^{(r)}$, which is isomorphic to the direct sum $(M \otimes_R S)^{(n)}$. Applying Lemma 0.1 to this direct sum decomposition defines the homomorphism $\Delta : \operatorname{Hom}_S(M \otimes_R S, M \otimes_R S) \to \operatorname{Hom}_S((M \oplus N) \otimes_R S, (M \oplus N) \otimes_R S)$. By [2, Proposition 7.1.10], $\operatorname{Hom}_R(M, M)$ is an Azumaya R-algebra, hence it is an R-progenerator [2, Theorem 7.1.4] and the natural map $\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(M, M) \otimes_R S$ is one-to-one. By [2, Proposition 2.4.1], $\operatorname{Hom}_R(M, M) \otimes_R S$ is isomorphic to $\operatorname{Hom}_S((M \oplus N) \otimes_R S, (M \oplus N) \otimes_R S)$. Similarly, the natural map $\operatorname{Hom}_R(M \oplus N, M \oplus N) \to \operatorname{Hom}_S((M \oplus N) \otimes_R S, (M \oplus N) \otimes_R S)$ is one-to-one. Consider the diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_S(M \otimes_R S, M \otimes_R S) & \xrightarrow{\ \Delta\ } & \operatorname{Hom}_S((M \oplus N) \otimes_R S, (M \oplus N) \otimes_R S) \\
\cong \Big\uparrow & & \Big\uparrow \cong \\
\operatorname{Hom}_R(M, M) \otimes_R S & & \operatorname{Hom}_R(M \oplus N, M \oplus N) \otimes_R S \\
\subseteq \Big\uparrow & & \Big\uparrow \subseteq \\
\operatorname{Hom}_R(M, M) & \overset{\exists \delta}{\dashrightarrow} & \operatorname{Hom}_R(M \oplus N, M \oplus N)
\end{array}
\tag{1}
$$

of homomorphisms of R-algebras. Next we show that Δ restricts to a homomorphism $\delta : \operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(M \oplus N, M \oplus N)$. The proof is by faithfully flat descent. Start with a basis $\{b_1, \dots, b_r\}$ for the S-module $M \otimes_R S$ and extend it to a basis for $(M \oplus N) \otimes_R S$. With respect to these bases, interpret $\operatorname{Hom}_S(M \otimes_R S, M \otimes_R S)$ as r-by-r matrices over S (denoted $M_r(S)$) and $\operatorname{Hom}_S((M \oplus N) \otimes_R S, (M \oplus N) \otimes_R S)$ as rn-by-rn matrices over S (denoted $M_{rn}(S)$). We see that $\Delta : M_r(S) \to M_{rn}(S)$ sends a matrix A to the block diagonal matrix $A \oplus \cdots \oplus A$. Let $e_0 : S \to S \otimes_R S$ be defined by $s \mapsto 1 \otimes s$. Likewise, let $e_1 : S \to S \otimes_R S$ be defined by $s \mapsto s \otimes 1$. Then each e_i is an R-algebra homomorphism. Let $\mathfrak{F}_i$ be the functor from S-modules to $S \otimes_R S$-modules induced by tensoring with e_i. From the description of Δ above we see that $\mathfrak{F}_0(\Delta)$ is equal to $\mathfrak{F}_1(\Delta)$. By faithfully flat descent ([2, Proposition 5.3.4]), there exists an R-algebra homomorphism δ such that diagram (1) commutes. By the homomorphism δ, we can view $\operatorname{Hom}_R(M, M)$ as a ring of endomorphisms of the R-module $M \oplus N$. By the Morita Theorem ([2, Theorem 1.5.2]), there is an R-module P and a left $\operatorname{Hom}_R(M, M)$-module isomorphism $\sigma : P \otimes_R M \to M \oplus N$. Since $\operatorname{Hom}_R(M, M)$ is an R-algebra, σ is an R-module isomorphism. Since $M \oplus N$ is a free R-module of rank $s = rn$, we are finished. □

References

[1] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR 0249491 (40 #2736)

[2] Timothy J. Ford, *Separable algebras*, Graduate Studies in Mathematics, vol. 183, American Mathematical Society, Providence, RI, 2017. MR 3618889

[3] Bernard R. McDonald, *Linear algebra over commutative rings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 87, Marcel Dekker, Inc., New York, 1984. MR 769104 (86d:13008)

Department of Mathematics, Florida Atlantic University, Boca Raton, Florida 33431

Email address: ford@fau.edu