COMMUNICATIONS IN ALGEBRA, 18(11), 3685–3687 (1990)

ON A REMARK OF GROTHENDIECK

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In [5, II] Grothendieck suggested the problem of finding a normal algebraic surface, such that the kernel of the map $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is non-trivial where K is the function field of X [5, page 75]

Tim Ford told us about a possible construction in characteristic zero. Some details still have to be worked out. Here we present an example in characteristic $p > 5$.

Let

$$W \, : \, z^p = \sum_{0 \le i+j \le p} T_{ij} x^i y^j$$

be a generic Zariski surface $p > 5$. It was shown in [1], [2] that

$$\mathrm{Cl}\,W = 0$$

Nevertheless, W is singular, it has $p^2 - 3p + 3$ singularities all rational and of type A_{p-1} [3]. Let ϕ be one such singularity. Let $\mathscr{O}_{W,\phi}$ be the local ring of ϕ in W, $X = \mathrm{Spec}\,\mathscr{O}_{W,\phi}$, $\mathrm{Cl}\,X = 0$.

Let $\mathscr{O}^h_{W,\phi}$ be the strict henselization of $\mathscr{O}_{W,\phi}$. Set

$$X^h = \mathrm{Spec}\,\mathscr{O}^h_{W,\phi}.$$

Then

$$\mathrm{Cl}\,X^h = \mathbb{Z}/p\mathbb{Z}$$

by Lipman [3].

Let $U = X - \{\phi\}$ and $U^h = X^h - \{\phi\}$. We consider the commutative diagram

(1)
$$\begin{array}{ccccccccc}
\operatorname{Pic} X & & \operatorname{Cl} X & & & & & & \\
\downarrow{=} & & \downarrow{=} & & & & & & \\
\mathrm{H}^1(X,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1(U,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_\phi(X,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2(X,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2(U,\mathbb{G}_m) \\
\downarrow & & \downarrow & & \cong\downarrow\gamma & & \downarrow & & \\
\mathrm{H}^1(X^h,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1(U^h,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_\phi(X^h,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2(X^h,\mathbb{G}_m) & & \\
\downarrow{=} & & \downarrow{=} & & & & \downarrow{=} & & \\
\operatorname{Pic} X^h = 0 & & \operatorname{Cl} X^h & & & & 0 & &
\end{array}$$

with exact rows. The top and bottom rows of (1) are exact by [4, III, 1.25]. The map γ in (1) is the excision isomorphism [4, III, 1.28].

Take a non-zero element $\alpha \in \operatorname{Cl} X^h$. Using (1), we obtain a non-zero element of $\operatorname{Br}(X)$ which is a subgroup of the torsion subgroup of $\mathrm{H}^2(X,\mathbb{G}_m)$. Moreover α maps to zero in $\operatorname{Br}(U)$ thus in $\operatorname{Br}(K)$. Hence the element α is in $\operatorname{Ker}\big(\operatorname{Br}(\mathscr{O}_{W,\phi}) \to \operatorname{Br}(K)\big)$.

$$\operatorname{Br}\big(\mathscr{O}_{W,\phi}\big) = \varinjlim_{\phi \in V \subset W} \operatorname{Br}(V)$$

Hence the element α lives in $\operatorname{Br}(V)$ for some V. This answers the question or rather mild challenge raised by Grothendieck in [5, II]. A similar example could be constructed in characteristic $p = 3$ using Lang [6]. The question remains open in characteristic two, and in characteristic zero.

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Received: September 1989
Revised: February 1990