GENERALIZATIONS TO NON-NOETHERIAN RINGS SUPPLEMENTAL NOTES ON "SEPARABLE ALGEBRAS" BY T. J. FORD GRADUATE STUDIES IN MATHEMATICS, VOL. 183, 2017 June 23, 2026

TIMOTHY J. FORD

In the following notes we extend some theorems from [1] to include the case where the ground ring is non-noetherian. In Corollary 1.11 we show that an open immersion is an algebra of finite presentation, which was stated without proof in [1, Example 9.2.4 (3)]. Lemma 1.12 is a local to global lemma for a homomorphism of finite presentation which generalizes [1, Lemma 2.1.5]. In Lemma 1.5 we prove [1, Example 9.2.2 (4) and (5)]. Theorem 2.1, a generalization of [1, Theorem 9.3.9], proves that locally an étale algebra is isomorphic to a standard étale algebra. Theorem 2.3, a generalization of [1, Proposition 9.2.9], proves that if S is an étale R-algebra, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is an open map. Theorem 3.6, which generalizes [1, Theorem 9.3.12], is a theorem of permanance for an étale algebra over a non-noetherian ground ring. The proofs below utilize only theorems proven in [3] and [1].

1. Algebras of Finite Presentation

One goal of this section is to prove that if $f : R \to S$ is a homomorphism of commutative rings and $f^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is an open immersion, then S is an R-algebra of finite presentation (see Corollary 1.11 below). This theorem is referenced without proof in [1, Example 9.2.4 (3)]. The corollary and its proof presented here are based on [4, Proposition I.4.6, p. 328]. We begin by proving some results on algebras of finite presentation.

Throughout, all rings are commutative. Let M be an R-module and α a nonzero element of R. Denote by M_α the localization of M with respect to the multiplicative set $\{1, \alpha, \alpha^2, \alpha^3, \dots\}$.

Definition 1.1. Let R be a commutative ring and M an R-module. We say M is *locally finitely generated*, if there exist elements $\alpha_1, \dots, \alpha_n$ in R such that $R = R\alpha_1 + \dots + R\alpha_n$ and for each i, M_{α_i} is a finitely generated R_{α_i}-module.

Lemma 1.2. *Let M be an R-module. If M is locally finitely generated, then M is finitely generated.*

Proof. Let $\alpha_1, \dots, \alpha_n$ in R such that $R = R\alpha_1 + \dots + R\alpha_n$ and for each i, M_{α_i} is a finitely generated R_{α_i}-module. For each i, let $\{x_{ij} \mid 1 \le j \le n_i\}$ be a subset of M which is a generating set for M_{α_i} as an R_{α_i}-module. The reader should verify that $\bigcup_i \{x_{ij} \mid 1 \le j \le n_i\}$ is a generating set for the R-module M. □

Definition 1.3. Let S be a commutative R-algebra with structure homomorphism $\theta : R \to S$. We say S is an *R-algebra of finite presentation*, if there is a polynomial ring in $n \geq 1$ variables over R and a surjective R-algebra homomorphism $\phi : R[x_1, \dots, x_n] \to S$ such that the kernel of ϕ is a finitely generated ideal. The R-algebra S is said to be *locally of finite presentation*, if there exist elements $\beta_1, \dots, \beta_n$ in S such that $S = S\beta_1 + \cdots + S\beta_n$ and for each i, S_{β_i} is an R-algebra of finite presentation. We also say θ is *locally of finite presentation*.

Example 1.4. Let α be a non-zero element of R. Define $\phi : R[x] \to R_\alpha$ by $\phi(x) = 1/\alpha$. The kernel of ϕ is the principal ideal generated by $x\alpha - 1$. This shows R_α is an R-algebra of finite presentation.

Lemma 1.5. *(*[1, Example 9.2.2 (4) and (5)]*)*

(1) *(Change of Base) Let $f : R \to T$ be a homomorphism of commutative rings such that T is an R-algebra of finite presentation. If $g : R \to S$ is any homomorphism of commutative rings, then $f \otimes 1 : S \to T \otimes_R S$ makes $T \otimes_R S$ into an S-algebra of finite presentation.*

(2) *(Finitely Presented over Finitely Presented is Finitely Presented) If S is a finitely presented R-algebra and T is a finitely presented S-algebra, then T is a finitely presented R-algebra.*

Proof. (1): There exists an onto homomorphism $\phi : R[x_1, \dots, x_n] \to T$ such that $\ker \phi$ is a finitely generated ideal. Then $\phi \otimes 1 : S[x_1, \dots, x_n] \to T \otimes_R S$ is onto. The kernel of $\phi \otimes 1$ is generated by the image of $\ker \phi \otimes_R S$, hence is a finitely generated ideal.

(2): There exist homomorphisms $\phi : R[x_1, \dots, x_m] \to S$ and $\psi : S[y_1, \dots, y_n] \to T$ such that ϕ and ψ are both onto, and $\ker \phi$ and $\ker \psi$ are both finitely generated ideals. Define $\chi : R[x_1, \dots, x_m][y_1, \dots, y_n] \to T$ by $\chi(x_i) = \psi\phi(x_i)$ for each i and $\chi(y_i) = \psi(y_j)$ for each j. Then χ is onto and the kernel of χ is generated by $\ker \phi + \ker \psi$, hence is finitely generated. □

Proposition 1.6. *Let $f : R \to S$ be a homomorphism of commutative rings. The following are true.*

(1) *If the kernel of f is a finitely generated ideal and f is onto, then S is an R-algebra of finite presentation.*

(2) *If f is one-to-one and S is a finitely generated R-module, then S is an R-algebra of finite presentation.*

(3) *If the kernel of f is a finitely generated ideal and S is a finitely generated R-module, then S is an R-algebra of finite presentation.*

Proof. The proof is left to the reader as an exercise. □

Proposition 1.7. (1) *If $f : R \to S$ is a local isomorphism, then f is locally of finite presentation.*

(2) *(Composition) If $f : R \to S$ and $g : S \to T$ are both locally of finite presentation, then $g \circ f : R \to T$ is locally of finite presentation.*

(3) *(Change of Base) Let $f : R \to T$ and $g : R \to S$. If f is locally of finite presentation, then $f \otimes 1 : S \to T \otimes_R S$ makes $T \otimes_R S$ into an S-algebra that is locally of finite presentation.*

(4) *(Tensor Product) Let $f : S \to S'$ and $g : T \to T'$ be R-algebra homomorphisms. If f and g are both locally of finite presentation, then $f \otimes g : S \otimes_R T \to S' \otimes_R T'$ is locally of finite presentation.*

(5) *(Descent) If $f : R \to S$ is locally of finite type and $g : S \to T$, and $g \circ f : R \to T$ is locally of finite presentation, then g is locally of finite presentation.*

Proof. (1): For every $\mathfrak{q} \in \operatorname{Spec} S$ there exists $\alpha \in R$ such that $f(\alpha) \in S - \mathfrak{q}$ and $R_\alpha \to S_\alpha$ is an isomorphism. By Example 1.4, S_α is an R-algebra of finite presentation. By [3, Exercise 3.3.29], S is locally of finite presentation.

(2): Let $\mathfrak{r} \in \operatorname{Spec} T$ be arbitrary. There exists $\beta \in T - \mathfrak{r}$ such that $g : S \to T_\beta$ is of finite presentation. Let $\mathfrak{q} = \mathfrak{r} \cap S$. There exists $\alpha \in S - \mathfrak{q}$ such that $f : R \to S_\alpha$ is of finite presentation. By Lemma 1.5 (1), $g : S_\alpha \to T_{\beta\alpha}$ is of finite presentation. By Lemma 1.5 (2), $g \circ f : R \to S_\alpha \to T_{\beta\alpha}$ is of finite presentation. Because $\operatorname{Spec} S$ is compact ([3, Exercise 3.3.29]), we see that T is locally of finite presentation as an R-algebra.

(3): Start with elements $\beta_1, \dots, \beta_n$ in T such that $T = T\beta_1 + \cdots + T\beta_n$ and for each i, T_{β_i} is an R-algebra of finite presentation. Then $\beta_1 \otimes 1, \dots, \beta_n \otimes 1$ generate the unit ideal in $T \otimes_R S$. By Lemma 1.5 (1), $T_{\beta_i} \otimes_R S$ is an S-algebra of finite presentation, for each i.

(4): By (3), $f \otimes 1 : S \otimes_R T \to S' \otimes_R T$ and $1 \otimes g : S' \otimes_R T \to S' \otimes_R T'$ are both locally of finite presentation. By (2), $f \otimes g$ is locally of finite presentation.

(5): First note that $S \otimes_{S \otimes_R S} (T \otimes_R S)$ is isomorphic to T. The diagram

$$\begin{tikzcd} & T & \\ T \otimes_R S \arrow[ur, "\rho"] & & S \arrow[ul, "g"'] \\ & S \otimes_R S \arrow[ul, "g\otimes 1"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

commutes, where $\rho(t \otimes s) = ts$. By [3, Proposition 3.5.14], f makes S into a finitely generated R-algebra. By [3, Lemma 10.1.6], the multiplication homomorphism $\mu : S \otimes_R S \to S$ makes S into a finitely presented $S \otimes_R S$-module. By Proposition 1.6, S is a finitely presented $S \otimes_R S$-algebra. By (3), ρ makes T into a $T \otimes_R S$-algebra of finite presentation. By (3) again, $(g \circ f) \otimes 1 : S \to T \otimes_R S$ is locally of finite presentation. The diagram

$$\begin{tikzcd} S \arrow[rr, "g"] \arrow[dr, "(g\circ f)\otimes 1"'] & & T \\ & T \otimes_R S \arrow[ur, "\rho"'] & \end{tikzcd}$$

commutes, where $\rho(t \otimes s) = ts$. By (2), g is locally of finite presentation. $\square$

Lemma 1.8. *Let I be a proper ideal in R and α a nonzero element of R such that R_α / I_α is an R-algebra of finite presentation. Then I_α is a finitely generated ideal in R_α.*

Proof. There exists an onto R-algebra homomorphism $\phi : R[x_1, \dots, x_n] \to R_\alpha / I_\alpha$ such that $\ker \phi$ is a finitely generated ideal. Tensor with $(\cdot) \otimes_R R_\alpha$ to get the exact sequence

$$0 \to \ker \phi \otimes_R R_\alpha \to R_\alpha[x_1, \dots, x_n] \xrightarrow{\tau} R_\alpha / I_\alpha \to 0$$

where $\tau = \phi \otimes 1$. Therefore, $\ker \tau = \ker \phi \otimes_R R_\alpha$ is a finitely generated ideal. Let $\eta : R_\alpha \to R_\alpha/I_\alpha$ be the natural map. Since τ is an R_α-algebra homomorphism, there exists a lifting $\psi : R_\alpha[x_1, \dots, x_n] \to R_\alpha$ such that $\eta\psi(x_i) = \tau(x_i)$ for each i. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\tau \arrow[r] \arrow[d] & R_\alpha[x_1,\dots,x_n] \arrow[r, "\tau"] \arrow[d, "\psi"] & R_\alpha/I_\alpha \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & I_\alpha \arrow[r] & R_\alpha \arrow[r, "\eta"] & R_\alpha/I_\alpha \arrow[r] & 0
\end{tikzcd}$$

commutes and the rows are exact sequences. Since ψ is onto and u is the identity map, the Snake Lemma ([3, Theorem 2.3.2]) implies $\psi(\ker\tau) = I_\alpha$. This proves I_α is a finitely generated ideal in R_α. □

Proposition 1.9. *Let $f : R \to S$ be a homomorphism of commutative rings. Let $\theta : R[x_1, \dots, x_n] \to S$ be an onto homomorphism of R-algebras. Then S is an R-algebra of finite presentation if and only if $\ker\theta$ is a finitely generated ideal.*

Proof. First assume f is of finite presentation. The diagram of R-algebra homomorphisms

$$\begin{tikzcd}
R[x_1,\dots,x_n] \arrow[rr, "\theta"] & & S \\
 & R \arrow[ul, "\sigma"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes, where σ is the natural map. Since σ is locally of finite type, by Proposition 1.7 (5), θ is locally of finite presentation. By Lemma 1.8, $\ker\theta$ is locally a finitely generated ideal in $R[x_1, \dots, x_n]$. By Lemma 1.2, $\ker\theta$ is a finitely generated ideal. The converse is immediate. □

Proposition 1.10. *If $f : R \to S$ is locally of finite presentation, then f is of finite presentation.*

Proof. There is an onto R-algebra homomorphism $\theta : R[x_1, \dots, x_n] \to S$. Let $\sigma : R \to R[x_1, \dots, x_n]$ be the natural map. By Proposition 1.7 (5), θ is locally of finite presentation. Proceed as in the proof of Proposition 1.9. □

Corollary 1.11. *Let $f : R \to S$ be a homomorphism of commutative rings. If $f^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is an open immersion, then S is an R-algebra of finite presentation.*

Proof. By Proposition 1.7 (1), $f : R \to S$ is locally of finite presentation. By Proposition 1.10, $f : R \to S$ is of finite presentation. □

Lemma 1.12 is a local to global lemma for a homomorphism of finite presentation which generalizes [3, Lemma 3.1.13] and [1, Lemma 2.1.5].

Lemma 1.12. *Let $f : A \to B$ be a homomorphism of commutative R-algebras. Assume f makes B into an A-algebra of finite presentation. If $W \subseteq R$ is a multiplicative set and $f_W : W^{-1}A \to W^{-1}B$ is an isomorphism of $W^{-1}R$-algebras, then there exists $w \in W$ such that $f_w : A_w \to B_w$ is an isomorphism of R_w-algebras.*

Proof. As an A-algebra, B is finitely generated. Say B is generated as an A-algebra by the elements $y_1, \dots, y_m$. By [3, Lemma 3.1.11] there exists $\beta \in W$ such that $\beta(Ry_1 + \cdots + Ry_m) \subseteq f(A)$. Then $f_\beta : A_\beta \to B_\beta$ is an onto homomorphism of R_β-algebras. By Lemma 1.5, f_β makes B_β into an A_β-algebra of finite presentation. By Lemma 1.8, $(\ker f)_\beta$ is a finitely generated ideal in A_β. Say $(\ker f)_\beta = A_\beta x_1 + \cdots + A_\beta x_n$. Since $(\ker f)_\beta \otimes_R W^{-1}R = \ker f \otimes_R W^{-1}R = (0)$, there exists $\alpha \in W$ such that $\alpha(Rx_1 + \cdots + Rx_m) = (0)$. Hence $\alpha(\ker f)_\beta = (0)$. If we set $w = \alpha\beta$, then $(A_\beta)_\alpha = A_w$, $(B_\beta)_\alpha = B_w$ and $f_w : A_w \to B_w$ is an isomorphism of R_w-algebras. □

2. Standard Étale Algebras

If S is an étale R-algebra, we show that S is locally isomorphic to a standard étale algebra. Theorem 2.1, which is a generalization of [1, Theorem 9.3.9], does not require the ground ring R to be noetherian. As an application, in Theorem 2.3 below we show that $\operatorname{Spec} S \to \operatorname{Spec} R$ is an open map.

Theorem 2.1. *Let R be a commutative ring, S a commutative R-algebra, $\mathfrak{q}$ a prime ideal in S, and $\mathfrak{p} = \mathfrak{q} \cap R$. If S is étale over R in a neighborhood of $\mathfrak{q}$, then there exist $f \in S - \mathfrak{q}$ and $h \in R - \mathfrak{p}$ such that $S[f^{-1}]$ is isomorphic to a standard étale $R[h^{-1}]$-algebra of the form $\left(R[h^{-1}][x]/(P)\right)[g^{-1}]$ where P is a monic polynomial in $R[h^{-1}][x]$ and the derivative P' is invertible in $\left(R[h^{-1}][x]/(P)\right)[g^{-1}]$.*

Proof. The proof is divided into four steps. As mentioned in [1, Remark 9.3.10], in the proof of [1, Theorem 9.3.9], the noetherian hypothesis on R was used for the local to global argument in Step 2. The noetherian hypothesis was also used in the last line of Step 4 to assume the kernel of h is a finitely generated ideal. For sake of completeness, we include all four steps.

Step 1: Reduction to the case where S is a finitely generated R-module. We assume S is R-separable and an R algebra of finite presentation. By [1, Exercise 9.3.5], S is quasi-finite over R. By Zariski's Main Theorem, [3, Corollary 7.4.16], there exists a factorization $R \to T \to S$ such that T is a finitely generated R-module and there exists $f \in T - \mathfrak{q} \cap T$ such that $T[f^{-1}] = S[f^{-1}]$. The reader should verify that T is étale over R in a neighborhood of $\mathfrak{q} \cap T$, and that if the theorem is true for T, it is true for S.

Step 2: Reduction to the case where R is local. We assume that S is a separable flat R-algebra of finite presentation. Assume the theorem is true for $R_\mathfrak{p}$. So there exist polynomials $f, g \in R_\mathfrak{p}[x]$ such that f is monic, f' is invertible in $(R_\mathfrak{p}[x]/(f))[g^{-1}]$, $c \in S_\mathfrak{p} - \mathfrak{q}S_\mathfrak{p}$, and there is an $R_\mathfrak{p}$-algebra isomorphism

$$\left(\frac{R_\mathfrak{p}[x]}{(f)}\right)[g^{-1}] \xrightarrow{\phi} S_\mathfrak{p}[c^{-1}].$$

The reader should verify that there is an element $\alpha \in R - \mathfrak{p}$ such that f and g come from $R[\alpha^{-1}][x]$, f' is invertible in $\left(R[\alpha^{-1}][x]/(f)\right)[g^{-1}]$, c comes from $S[\alpha^{-1}]$, $\phi(x)$ comes from $S[\alpha^{-1}][c^{-1}]$, and $\phi(g^{-1})$ comes from a unit in $S[\alpha^{-1}][c^{-1}]$. Therefore, the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad\theta\quad} & S \\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow{\scriptstyle s} \\ \left(\frac{R[\alpha^{-1}][x]}{(f)}\right)[g^{-1}] & \xrightarrow{\quad\psi\quad} & S[\alpha^{-1}][c^{-1}] \\ \Big\downarrow & & \Big\downarrow \\ \left(\frac{R_{\mathfrak{p}}[x]}{(f)}\right)[g^{-1}] & \xrightarrow{\quad\phi\quad} & S_{\mathfrak{p}}[c^{-1}] \end{array}$$

commutes. The vertical maps r and s are étale and the map ψ is an $R[\alpha^{-1}]$-algebra homomorphism. The $R_{\mathfrak{p}}$-algebra isomorphism ϕ is induced by tensoring ψ with $R_{\mathfrak{p}}$. By [3, Proposition 10.5.12 (4)], ψ is of finite presentation. We can pass from the local to the global, by [3, Proposition 10.5.15 and Lemma 10.5.16]. That is, there exists an element $\beta \in R - \mathfrak{p}$ such that

$$\left(\frac{R[\beta^{-1}][x]}{(f)}\right)[g^{-1}] \xrightarrow{\phi} S[\beta^{-1}][c^{-1}]$$

is an isomorphism of $R[\beta^{-1}]$-algebras. So if the theorem is true for $R_{\mathfrak{p}}$, it is true for R.

Step 3: Reduction to the case where $S = R[x]$ is a cyclic extension. By the previous steps, we assume R is local, S is separable over R, and finitely generated as an R-module. By [1, Theorem 8.3.7], $\mathfrak{p}$ generates the maximal ideal of $S_{\mathfrak{q}}$, and the extension of residue fields $k_{\mathfrak{p}} \to k_{\mathfrak{q}}$ is separable. Let $\bar{S} = S \otimes_R k_{\mathfrak{p}}$. Then $\bar{S}$ is artinian and the prime ideals of $\bar{S}$ correspond to those primes of S lying over $\mathfrak{p}$. Let $\bar{\mathfrak{q}}$ be the maximal ideal of $\bar{S}$ corresponding to $\mathfrak{q}$. Then $(\bar{S})_{\bar{\mathfrak{q}}} = k_{\mathfrak{q}}$. By the Primitive Element Theorem, $k_{\mathfrak{q}} = k_{\mathfrak{p}}(t)$ is a simple extension. By [3, Theorem 4.4.13], there is $x \in S$ which maps to t in $(\bar{S})_{\bar{\mathfrak{q}}}$, and such that x restricts to 0 for all other prime ideals of $\bar{S}$. Let $C = R[x]$, and $\mathfrak{r} = \mathfrak{q} \cap C$. Then S is a finitely generated C-module and $\mathfrak{q}$ is the only prime ideal of S lying over $\mathfrak{r}$. Thus $S_{\mathfrak{r}} = S_{\mathfrak{q}}$ is a finitely generated $C_{\mathfrak{r}}$-module. The composite map

$$k_{\mathfrak{p}}(t) \to C_{\mathfrak{r}}/\mathfrak{r}C_{\mathfrak{r}} \to S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} = k_{\mathfrak{q}}$$

is onto, hence $C_{\mathfrak{r}} \to S_{\mathfrak{q}}$ is onto ([3, Exercise 3.2.4]). Since $C \to S$ is one-to-one, by [3, Lemma 3.1.11], there is $f \in C - \mathfrak{r}$ such that $C[f^{-1}] = S[f^{-1}]$.

Step 4: Assume R is local, $S = R[x]$ is finitely generated as an R-module, $\mathfrak{q} \in \operatorname{Spec} S$ is the only prime lying above the maximal ideal $\mathfrak{p} \in \operatorname{Spec} R$, and S is separable, flat and finitely presented over R. So $S/\mathfrak{p}S$ is an artinian local ring, and is separable over the field $k_{\mathfrak{p}}$. It follows from [3, Corollary 8.5.6] that $S/\mathfrak{p}S$ is equal to the field $k_{\mathfrak{q}}$. Let n be the dimension of $k_{\mathfrak{q}}$ over $k_{\mathfrak{p}}$. Then $1, x, \dots, x^{n-1}$ generate $S/\mathfrak{p}S = k_{\mathfrak{q}}$ over $R/\mathfrak{p} = k_{\mathfrak{p}}$. By [3, Corollary 3.4.1] (Nakayama's Lemma), the R-module S is generated by $1, x, \dots, x^{n-1}$. There is a monic polynomial $P \in R[X]$ such that $h : R[X]/(P) \to S$ is onto. The natural map $\bar{h} : (R/\mathfrak{p})[X]/(P) \to S/\mathfrak{p}S$ is an isomorphism. The image of P in $k_{\mathfrak{p}}[X]$ is therefore the irreducible polynomial for t over $k_{\mathfrak{p}}$, where t is as in Step 3. Since $k_{\mathfrak{q}}$ is separable over $k_{\mathfrak{p}}$, (P, P') is the unit ideal in $k_{\mathfrak{p}}[X]$. Let $\mathfrak{q}' = h^{-1}(\mathfrak{q})$. By [1, Theorem 8.1.24], $(R[X]/(P))_{\mathfrak{q}'}$ is separable over R. By [3, Proposition 8.6.1], (P, P') is the unit ideal in $(R[X]/(P))_{\mathfrak{q}'}$. By [3, Lemma 3.1.11], there is some $b \in (R[X]/(P)) - \mathfrak{q}'$ such that $(R[X]/(P))[b^{-1}]$ is a standard étale R-algebra. If we set $c = h(b)$, then $h : (R[X]/(P))[b^{-1}] \to S[c^{-1}]$ is onto. By [1, Corollary 9.2.6], $h : (R[X]/(P))[b^{-1}] \to S[c^{-1}]$ is étale. The kernel of h is finitely generated, by [3, Lemma 10.5.13]. By [3, Exercise 3.2.4], h is a localization. □

Lemma 2.2 and its proof are based on [5, Chapitre V, lemma, p. 57].

Lemma 2.2. *Let S be an R-algebra such that S is an R-progenerator module of constant rank r. If $\theta : R \to S$ is the structure homomorphism, then the continuous map $\theta^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is open.*

Proof. Let $\alpha \in S - (0)$. We show that $\theta^\sharp(U(\alpha))$ is an open subset of $\operatorname{Spec} R$. By [1, Example 5.3.3], the characteristic polynomial of α over R exists. Let $\operatorname{char.poly}_R(\alpha) = x^r + a_{r-1}x^{r-1} + \cdots + a_1x + a_0$ in $R[x]$. Let $\mathfrak{q} \in U(\alpha)$. Then $\alpha \notin \mathfrak{q}$. Consider $\mathfrak{p} = \mathfrak{q} \cap R$. Let $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ be the residue field and $S \otimes_R k(\mathfrak{p})$ the fiber over $\mathfrak{p}$. Then $\mathfrak{q}$ is the image of the prime $\mathfrak{q} \otimes_R k(\mathfrak{p})$ under the map $\epsilon^\sharp : \operatorname{Spec} S \otimes_R k(\mathfrak{p}) \to \operatorname{Spec} S$. Since $\alpha \otimes 1 \notin \mathfrak{q} \otimes_R k(\mathfrak{p})$, this implies $\alpha \otimes 1$ is not in the nil radical $\operatorname{Rad}_{S\otimes_R k(\mathfrak{p})}(0)$. Therefore, the characteristic polynomial $\operatorname{char.poly}_{k(\mathfrak{p})}(\alpha)$ is not equal to x^r. In other words, for some i, $a_i \notin \mathfrak{p}$, which implies $\mathfrak{p}$ is in the open set $\operatorname{Spec} R - V(a_0, a_1, \ldots, a_{r-1})$. Conversely, if $\mathfrak{p}$ is in $\operatorname{Spec} R - V(a_0, a_1, \ldots, a_{r-1})$, then $\operatorname{char.poly}_{k(\mathfrak{p})}(\alpha)$ is not equal to x^r. Then $U(\alpha) \cap \operatorname{Spec} S \otimes_R k(\mathfrak{p}) \neq \emptyset$. There is some $\mathfrak{q} \in U(\alpha)$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. □

Theorem 2.3 is a generalization of [1, Proposition 9.2.9] for a non-noetherian base ring R.

Theorem 2.3. *If S is an étale R-algebra with structure homomorphism $\theta : R \to S$, then $\theta^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is an open map.*

Proof. Let $\mathfrak{q} \in \operatorname{Spec} S$. We need to show that there is an open neighborhood of $\mathfrak{q}$ which maps to an open neighborhood of $\mathfrak{p} = \mathfrak{q} \cap R$ in $\operatorname{Spec} R$. By Theorem 2.1, we can assume S is a standard étale algebra of the form $(R[x]/(P))\,[g^{-1}]$, where P is monic of degree r in $R[x]$. By Lemma 2.2, $\operatorname{Spec} R[x]/(P) \to \operatorname{Spec} R$ is an open map. The open immersion $\operatorname{Spec}(R[x]/(P))\,[g^{-1}] \to \operatorname{Spec} R[x]/(P)$ is open. The proof follows. □

3. Theorems of Permanance for Étale Algebras

If S is an étale R-algebra, we prove that S inherits some important properties of R. For instance, in Theorem 3.6 we show that if R is reduced, then S is reduced, and if R is normal, then S is normal. The proof is based on Theorem 2.1 and the analogous theorems of permanence for an étale R-algebra of the form $R[x]/(P)$ where P is a separable polynomial. The following results are based on [5, Chapitre VII, §1 and §2, pp. 72 – 75].

Let R be a commutative ring and $f(x) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + x^n$ a monic polynomial in $R[x]$ of degree n. Let $S = R[x]/(f(x))$. As in the proof of [1, Proposition 4.6.1], in $R[x, y]$ there is a polynomial $q(x, y)$, the divided difference, such that $f(y) - f(x) = (y - x)q(x, y)$. Expanding q as a polynomial in y, $q(x, y) = \sum_{j=0}^{n-1} q_j(x)y^j$ for $q_j(x) \in R[x]$. From the divided difference we get the derivative

$$f'(x) = q(x, x) = \sum_{j=0}^{n-1} q_j(x)x^j. \tag{3.1}$$

In $S[y]$ we have

$$f(y) = (y-x)q(x,y) = (y-x)\sum_{j=0}^{n-1} q_j(x)y^j. \tag{3.2}$$

The R-module S is free with basis $\{1, x, \dots, x^{n-1}\}$. Let π_i be the R-module projection of S onto the coefficient of x^i. Then $\{(x_i, \pi_i) \mid i = 0, \dots, n-1\}$ is a dual basis for S and the trace map is given by $T_R^S(z) = \sum_{i=0}^{n-1} \pi_i(zx^i)$.

Proposition 3.1 and its corollary are attributed to John Tate.

Proposition 3.1. *In the above context, the following are true.*

(1) $\{(x_i, q_i(x)\pi_{n-1}) \mid i = 0, \dots, n-1\}$ *is a dual basis for* S.

(2) $T_R^S(z) = f'(x)\pi_{n-1}(z)$.

Proof. First we establish some notation. Tensoring $\pi_{n-1} : S \to R$ with $R[y]$, we have the $R[y]$-module homomorphism $\pi_{n-1} \otimes 1 : S[y] \to R[y]$. For simplicity in the following, we will simply write π_{n-1} for both maps. The R-module homomorphism $\sigma_0 : S \to R[y]$ is defined on the free basis $\{x^i \mid i = 0, \dots, n-1\}$ by the rule $\sigma_0(x^i) = y^i$. We define $\sigma : S[y] \to R[y]$ by $\sigma(\sum_{j=0}^{p} s_j y^j) = \sum_{j=0}^{p} \sigma_0(s_j)y^j$. Let $h(y)$ be an arbitrary polynomial in $S[y]$. We show

$$f(y)\pi_{n-1}(h(y)) = \sigma((y-x)h(y)). \tag{3.3}$$

Because π_{n-1} and σ are $R[y]$-linear, it suffices to prove Eq. (3.3) when $h(y)$ is one of the basis elements $x^i \in S$. There are two cases. If $0 \le i < n-1$, then $\pi_{n-1}(x^i) = 0$, so the left hand side is 0. The right hand side is $\sigma((y-x)x^i) = \sigma(yx^i - x^{i+1}) = \sigma_0(x^i)y - \sigma_0(x^{i+1}) = y^iy - y^{i+1} = 0$. In this case, Eq. (3.3) is true. Now when $h(y) = x^{n-1}$, the left hand side of Eq. (3.3) is $f(y)$. The right hand side is

$$\begin{aligned} \sigma((y-x)x^{n-1}) &= \sigma(yx^{n-1} + r_{n-1}x^{n-1} + \cdots + r_1x + r_0) \\ &= y^n + r_{n-1}y^{n-1} + \cdots + r_1x + r_0) \\ &= f(y) \end{aligned}$$

since $f(x) = 0$ in S. Therefore Eq. (3.3) holds in every case.

For $h(y) = x^i \sum_{j=0}^{n-1} q_j(x)y^j$, Eq.(3.3) becomes

$$\begin{aligned} f(y)\sum_{j=0}^{n-1} \pi_{n-1}\left(x^iq_j(x)\right)y^j &= \sigma\left((y-x)x^i\sum_{j=0}^{n-1} q_j(x)y^j\right) \\ &= \sigma\left(x^if(y)\right) \\ &= y^if(y) \end{aligned}$$

where we used Eq.(3.2) to get the second equation from the first. Since $f(y)$ is monic it is not a divisor of zero. Cancelling,

$$\sum_{j=0}^{n-1} \pi_{n-1}\left(x^iq_j(x)\right)y^j = y^i$$

which implies $\pi_{n-1}\left(x^iq_j(x)\right) = \delta_{ij}$ (Kronecker delta). Therefore, $q_j(x)\pi_{n-1}$ is equal to π_j. This proves (1).

The trace formula in (2) follows from:

$$T_R^S(z) = \sum_{i=0}^{n-1} \pi_i(zx^i) = \sum_{i=0}^{n-1} \pi_{n-1}(q_i(x)zx^i) = \sum_{i=0}^{n-1} q_i(x)x^i\pi_{n-1}(z) = f'(x)\pi_{n-1}(z)$$

where the last equation is from Eq.(3.1). □

Corollary 3.2. *For every* $z \in S$, $f'(x)z = \sum_{i=0}^{n-1} T_R^S(q_i(x)z)x^i$.

Proof. Using the dual basis of Proposition 3.1 (1) for the R-module S, we have

$$f'(x)z = \sum_{i=0}^{n-1} \pi_{n-1}(q_i(x)f'(x)z)x^i.$$

It follows from Proposition 3.1 (2) that

$$\pi_{n-1}(q_i(x)f'(x)z) = f'(x)\pi_{n-1}(q_i(x)z) = T_R^S(q_i(x)z).$$

□

Lemma 3.3. *Let R be a commutative ring, $f(x)$ a monic polynomial of degree n in $R[x]$, and $S = R[x]/(f(x))$. Then*

$$f'(x)\operatorname{Rad}_S(0) \subseteq \operatorname{Rad}_R(0)S.$$

Proof. Let $P \in \operatorname{Spec} R$ and $k(P) = R_P/(PR_P)$, the residue field at P. Consider the commutative diagram

$$\begin{tikzcd} S \arrow[r, "1\otimes\eta"] & S \otimes_R k(P) \\ R \arrow[u] \arrow[r, "\eta"] & k(P) \arrow[u] \end{tikzcd}$$

As noted in [3, Definition 8.6.5], $T_{k(P)}^{S\otimes_R k(P)} = T_R^S \otimes 1$. If z is a nilpotent element in S, then $z \otimes 1$ is a nilpotent element in $S \otimes_R k(P)$. By [2, Exercise 6.2.31], the trace of $z \otimes 1$ is 0 in $k(P)$. Since $k(P)$ is equal to the quotient field of R/P, the kernel of the natural map $\eta : R \to k(P)$ is P. Therefore, $T_R^S(z)$ is in P. Since P was arbitrary, this proves $T_R^S(z) \in \operatorname{Rad}_R(0)$. By Corollary 3.2,

$$f'(x)z = \sum_{i=0}^{n-1} T_R^S(q_i(x)z)x^i \in \sum_{i=0}^{n-1} \operatorname{Rad}_R(0)x^i \subseteq \operatorname{Rad}_R(0)S$$

since $q_i(x)z \in \operatorname{Rad}_S(0)$ for each i. □

Let R be a commutative ring and $f(x)$ a monic polynomial of degree n in $R[x]$. If $R[x]/(f(x))$ is a separable R-algebra, then we say $f(x)$ is a *separable polynomial*.

Theorem 3.4. *Let R be a commutative ring such that* $\operatorname{Rad}_R(0) = (0)$. *If $f(x)$ is a monic polynomial in $R[x]$ and $S = R[x]/(f(x))$, then* $\operatorname{Rad}_{S_{f'}}(0) = (0)$. *If $f(x)$ is separable, then* $\operatorname{Rad}_S(0) = (0)$.

Proof. By Lemma 3.3, $f'(x)\operatorname{Rad}_S(0) = (0)$. By [3, Exercise 3.3.28], upon localizing we have $f'(x)\operatorname{Rad}_{S_{f'}}(0) = (0)$. Since $f'(x)$ is invertible in $S_{f'}$, $\operatorname{Rad}_{S_{f'}}(0) = (0)$. If $f(x)$ is separable, then $S = S_{f'}$ ([3, Proposition 8.6.1]). □

Theorem 3.5. *Let R be an integrally closed integral domain, $f(x)$ a monic polynomial in $R[x]$ of degree n, and $S = R[x]/(f(x))$. Then $S_{f'}$ is a normal ring. If $f(x)$ is separable, then S is normal.*

Proof. Let K denote the quotient field of R. Set $\Lambda = S \otimes_R K \cong K[x]/(f(x))$ and let $\bar{S}$ denote the integral closure of S in Λ. The diagram

$$\begin{tikzcd} S \arrow[r] & \bar{S} \arrow[r] & \Lambda = S \otimes_R K \\ R \arrow[u] \arrow[rr] & & K \arrow[u] \end{tikzcd}$$

commutes, every arrow is one-to-one. Consider the trace mapping $T_K^\Lambda : \Lambda \to K$. Since $\bar{S}$ is integral over R and R is integrally closed in K, if α is an element of $\bar{S}$, then by [3, Theorem 5.1.11], $T_K^\Lambda(\alpha)$ is an element of R. By Corollary 3.2 applied to $\alpha \in \Lambda$,

$$f'(x)\alpha = \sum_{i=0}^{n-1} T_K^\Lambda(q_i(x)\alpha)x^i$$

where x denotes the coset of x in $S = R[x]/(f(x))$. Because each $q_i(x)$ is in S, we have $q_i(x)\alpha \in \bar{S}$. Hence $T_K^\Lambda(q_i(x)\alpha) \in R$. This proves $f'(x)\alpha \in S$, which implies $\bar{S} \subseteq S_{f'}$. This shows $S_{f'}$ is equal to $(\bar{S})_{f'}$. By [3, Lemma 5.1.7], $(\bar{S})_{f'}$ is the integral closure of $S_{f'}$ in $S_{f'} \otimes_R K$. Since Λ is finite dimensional over K, Λ is artinian. By Theorem 3.4, the nilradical of $S_{f'}$ is trivial. Since $S_{f'} \otimes_R K$ is a localization of $S_{f'}$, the ring $S_{f'} \otimes_R K$ is semisimple. Since $S_{f'}$ is integrally closed in the semisimple ring $S_{f'} \otimes_R K$, by [3, Lemma 11.1.6], $S_{f'}$ is normal. □

Theorem 3.6 is a generalization of [1, Theorem 9.3.12] for non-noetherian rings.

Theorem 3.6. *Let S be an étale R-algebra.*

(1) *For every $\mathfrak{q} \in \operatorname{Spec} S$, if $\mathfrak{p} = \mathfrak{q} \cap R$, then $\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p})$.*
(2) *If R is regular, then S is regular.*
(3) *If R is a normal ring, then S is a normal ring.*
(4) *If $\operatorname{Rad}_R(0) = (0)$, then $\operatorname{Rad}_S(0) = (0)$. In other words, if R is reduced, then S is reduced.*

Proof. Let $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$. By Theorem 2.1, there exist $\beta \in S - \mathfrak{q}$ and $\alpha \in R - \mathfrak{p}$ such that S_β is isomorphic to a standard étale R_α-algebra of the form $(R_\alpha[x]/(P))[g^{-1}]$ where P is a monic polynomial in $R_\alpha[x]$ and P' is invertible in $(R_\alpha[x]/(P))[g^{-1}]$.

(4): Since $\operatorname{Rad}_R(0) = (0)$, it follows that $\operatorname{Rad}_{R_\alpha}(0) = (0)$. By Theorem 3.4, $\operatorname{Rad}_{S_\beta}(0) = (0)$. Since $\mathfrak{q}$ was arbitrary, there is an open cover of $\operatorname{Spec} S$ by basic open sets of the form $U(\beta)$, where S_β is reduced. From this it follows that S is reduced.

(3): Since R is normal, it follows that R_α is normal. By Theorem 3.5, S_β is normal. Therefore, $S_\mathfrak{q}$ is an integrally closed integral domain. Since $\mathfrak{q}$ was arbitrary, S is normal.

(1): By [3, Exercise 8.6.24], S is a quasi-finite R-algebra. By Zariski's Main Theorem, [3, Corollary 7.4.16], there exists a factorization $R \to T \to S$ such that T is finitely generated as an R-module and $\operatorname{Spec} S \to \operatorname{Spec} T$ is an open immersion. Let $\mathfrak{r} = \mathfrak{q} \cap T$. Then $T_\mathfrak{r} \cong S_\mathfrak{q}$. It suffices to show $\dim R_\mathfrak{p} = \dim T_\mathfrak{r}$. Let $T_\mathfrak{p} = T \otimes_R R_\mathfrak{p}$. Then $T_\mathfrak{p}$ is finitely generated as an $R_\mathfrak{p}$-module. Since $T_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-algebra and the prime ideal $\mathfrak{r} \otimes_R R_\mathfrak{p}$ in $T_\mathfrak{p}$ lies over the maximal ideal $\mathfrak{p}R_\mathfrak{p}$, by [3, Lemma 3.5.5], $T_\mathfrak{p}$ is faithfully flat over $R_\mathfrak{p}$. It follows from [3, Theorem 5.3.6 (3)] that a chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n$ of length n in $\operatorname{Spec}(T_\mathfrak{p})$ gives rise to a chain $Q_0 \cap R_\mathfrak{p} \subsetneq Q_1 \cap R_\mathfrak{p} \subsetneq \cdots \subsetneq Q_n \cap R_\mathfrak{p}$ of length n in $\operatorname{Spec}(R_\mathfrak{p})$. Thus $\dim(T_\mathfrak{p}) \le \dim(R_\mathfrak{p})$. To finish the proof, it suffices to show that the height of $\mathfrak{r} \otimes_R R_\mathfrak{p}$ is greater than or equal to $\dim R_\mathfrak{p}$. By [3, Proposition 5.3.4] applied iteratively, a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq \mathfrak{p}R_\mathfrak{p}$ of length n in $\operatorname{Spec}(R_\mathfrak{p})$ lifts to a chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq \mathfrak{r} \otimes_R R_\mathfrak{p}$ of length n in $\operatorname{Spec}(T_\mathfrak{p})$.

(2): By hypothesis, $R_\mathfrak{p}$ is a noetherian regular local ring. Say $\dim(R_\mathfrak{p}) = d$. Then $S_\mathfrak{q}$ is a finitely generated $R_\mathfrak{p}$-algebra, hence is noetherian, by the Hilbert Basis Theorem. By (1), $\dim(S_\mathfrak{q}) = d$. The maximal ideal of $R_\mathfrak{p}$ is generated by a regular system of parameters $\{\pi_1, \dots, \pi_d\}$. Since $S_\mathfrak{q}$ is separable over $R_\mathfrak{p}$, $\{\pi_1, \dots, \pi_d\}$ is a generating set for the maximal ideal of $S_\mathfrak{q}$ ([3, Exercise 8.5.9]). This shows $S_\mathfrak{q}$ is a regular local ring. $\square$

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Department of Mathematics, Florida Atlantic University, Boca Raton, Florida 33431

Email address: `ford@fau.edu`