ERRATA TO "SEPARABLE ALGEBRAS" BY T. J. FORD GRADUATE STUDIES IN MATHEMATICS, VOL. 183, 2017 September 8, 2025

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(09/07/2025) Change Proposition 1.3.4 and its proof to:

Proposition 1.3.4. *Let R be a commutative ring and let M and N be R-modules such that $M \otimes_R N$ is an R-generator module. Then the following are true.*

(1) *M and N are both R-generator modules.*
(2) *If $M \otimes_R N$ is R-projective, then M and N are both R-projective.*
(3) *If $M \otimes_R N$ is finitely generated as an R-module, then M and N are both finitely generated as R-modules.*
(4) *If $M \otimes_R N$ is an R-progenerator, then M and N are both R-progenerators.*

Proof. (1): By Exercise 1.1.11 there is a free R-module F_1 of finite rank and a surjective R-module homomorphism $f_1 : F_1 \otimes_R (M \otimes_R N) \to R$. By Lemma 1.1.4 there is a free R-module F_2 and a surjective R-module homomorphism $f_2 : F_2 \to M$. By Lemma 1.3.1, $1 \otimes f_2 : (F_1 \otimes_R M) \otimes_R F_2 \to (F_1 \otimes_R M) \otimes_R N$ is surjective. Lemma 1.3.1 shows that $F_1 \otimes_R M \otimes_R F_2$ is a direct sum of copies of M. Then $f_1 \circ (1 \otimes f_2)$ maps a direct sum of copies of M onto R. By another application of Exercise 1.1.11, M is an R-module generator. The same is true for N, by Lemma 1.3.1 (5).

(2) and (3): By Part (1) and Exercise 1.1.11 there is a surjective R-module homomorphism $f : M^{(n)} \to R$. But f is split since R is projective over R. By Exercise 1.3.7,

$$M^{(n)} \otimes_R N \xrightarrow{1 \otimes f} N \to 0$$

is split exact. If $M \otimes_R N$ is projective, then by Lemma 1.3.1 and Exercise 1.1.10, N is projective. If $M \otimes_R N$ is finitely generated, then so is N. The same conclusions apply to M, by Lemma 1.3.1 (5).

(4): Follows from (1), (2) and (3). □

(09/07/2025) On p. 275, in the proof of Proposition 7.8.1, in the line that begins "Conversely, assume the isomorphisms σ and τ are given."

change: "By Proposition 1.3.4, M is both ..." to: "The proof of Proposition 1.3.4 can be adapted to show that M is both ..."

(05/29/2025) On page 568, on line 10 of the proof of Theorem 14.1.9, Part (1),

change: "Lemma 7.1.1" to: "Corollary 7.1.9"

(05/28/2025) As mentioned below in the errata dated (07/31/2023), the proof of Theorem 14.1.9 (3) is incomplete. To correct the proof, insert the following example and corollary in Section 12.4, and the following proof of Theorem 14.1.9 (3).

Example 12.4.6. Let S be a commutative ring, A an S-algebra, G a finite group of automorphisms of S, and R a subring of S^G. Suppose G acts as a group of inner automorphisms of A. That is, suppose $\theta : G \to A^*$ is a homomorphism of groups. For each $\sigma \in G$, write θ_σ instead of $\theta(\sigma)$. Then for all σ, τ in G, $\theta_{\sigma\tau} = \theta_\sigma\theta_\tau$. So θ induces a homomorphism $G \to \operatorname{Inn}(A)$. Using θ we make A into a left $\Delta(S/R, G, 1)$-module. If $\Delta(S/R, G, 1) = \bigoplus_{\sigma\in G} Su_\sigma$, then for all x in A and s in S, we have $(su_\sigma)x = s\left(\theta_\sigma x \theta_\sigma^{-1}\right)u_\sigma$.

Corollary 12.4.7. *Let S/R be a Galois extension with finite group G. Let A be an S-algebra. Assume G acts on A as a group of R-algebra automorphisms and that the action is S-semilinear. Let $B_1 = A^G$ be the fixed ring under this G-action. As in Example 12.4.6, let $\theta : G \to A^*$ be a homomorphism of groups. Using θ and the first G-action, define a second G-action on A by the rule: $\sigma \cdot x = \theta_\sigma \sigma(x)\theta_\sigma^{-1}$, for all $\sigma \in G$ and $x \in A$. Let $B_2 = A^G$ be the the fixed ring under this second G-action. Then B_1 and B_2 are Brauer equivalent.*

Proof. Consider the R-algebra

$$\begin{aligned}\Delta_1 &= B_1 \otimes_R \Delta(S/R, G, 1)\\ &= A^G \otimes_R \bigoplus_{\sigma\in G} Su_\sigma\\ &= \bigoplus_{\sigma\in G}(A^G \otimes_R S)u_\sigma\\ &= \bigoplus_{\sigma\in G} Au_\sigma\end{aligned}$$

where $u_\sigma x = \sigma(x)u_\sigma$ for all $\sigma \in G$ and $x \in A$. For the second G-action on A, we repeat this construction. To distinguish them, we use v_σ for the basis elements in the trivial crossed product.

$$\Delta_2 = B_2 \otimes_R \Delta(S/R, G, 1) = \bigoplus_{\sigma\in G}(A^G \otimes_R S)v_\sigma = \bigoplus_{\sigma\in G} Av_\sigma$$

where $v_\sigma x = \theta_\sigma\sigma(x)\theta_\sigma^{-1}v_\sigma$ for all $\sigma \in G$ and $x \in A$. Define $\phi : \Delta_2 \to \Delta_1$ by mapping the basis element v_σ to $\theta_\sigma u_\sigma$. Since $\{\theta_\sigma \mid \sigma \in G\}$ are units in A, ϕ is a bijection. To see that ϕ is multiplicative, note that for each $x \in A$ we have

$$(\theta_\sigma u_\sigma)x = \left(\theta_\sigma\sigma(x)\theta_\sigma^{-1}\right)(\theta_\sigma u_\sigma).$$

From this we see that ϕ is an R-algebra isomorphism. Since S/R is G-Galois, $\Delta(S/R, G, 1)$ is a trivial R-Azumaya algebra. Hence B_1 is Brauer equivalent to B_2. □

Proof of Theorem 14.1.9 (3). Since A is R-Azumaya, $T^n_R(A)$ is R-Azumaya. Let $A_S = A \otimes_R S$. By the proof of Part (1), $T^n_R(A_S)$ is Azumaya over $T^n_R(S)$ and $T^n_R(A_S)f$ is Azumaya over $T^n_R(S)f$. Since S/R is a Galois extension, by Theorem 7.6.1, $T^n_R(A) \to T^n_R(A_S)$ is one-to-one. Therefore we identify $T^n_R(A)$ with its image in $T^n_R(A)f \subseteq T^n_R(A_S)f \subseteq T^n_R(A_S)$. By change of base, $T^n_R(A) \otimes_R T^n_R(S)f$ is an Azumaya $T^n_R(S)f$-algebra. The homomorphism

$$T^n_R(A) \otimes_R T^n_R(S)f \to T^n_R(A_S)f$$

of Exercise 1.3.8 is one-to-one. Counting the ranks on both sides shows that the map is onto. The G-action on $T^n_R(A_S)f$ is the extension of the G-action on $T^n_R(S)f$. On $T^n_R(A)$, the group G permutes the factors in the tensor algebra. By Theorem 11.2.2, G acts as a group of inner automorphisms of $T^n_R(A)$. We are in the context of Corollary 12.4.7. Hence $\left(T^n_R(A) \otimes_R T^n_R(S)f\right)^G = T^n_R(A)$ is Brauer equivalent to $\operatorname{Cor}^S_R(A_S)$. □

(05/28/2025) As mentioned below in the errata dated (07/31/2023), the proof of Theorem 14.1.8 (3) is incomplete. At the end of the proof of Theorem 14.1.8 (3), insert the following:

Proof of Theorem 14.1.8 (3). Let M be an invertible R-module and write M_S for $M \otimes_R S$. As in Definition 14.1.1, $T^n_R(S)f$ is Galois over R with group $G = \Sigma_n$. The R-module $T^n_R(M_S)f$ is projective of rank $n!$. The G-module action on $T^n_R(S)f$ extends to a G-action on $T^n_R(M_S)f$. Now consider the R-module $T^n_R(M) \otimes_R T^n_R(S)f$, which is also projective of rank $n!$. Define

$$T^n_R(M) \otimes_R T^n_R(S)f \xrightarrow{\theta} T^n_R(M_S)f$$

by mapping a typical generator $(x_1 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes \cdots \otimes y_n)f$ in the left hand side to $((x_1 \otimes y_1) \otimes \cdots \otimes (x_n \otimes y_n))\, f$ in $T^n_R(M_S)f$. Then θ is an R-module homomorphism. Since θ is onto, θ is an isomorphism. The G-module action on $T^n_R(M_S)f$ induces a G-module action on $T^n_R(M) \otimes_R T^n_R(S)f$. On the factor $T^n_R(S)f$, it is the usual G-action which permutes the factors in the tensor product. Likewise, on the factor $T^n_R(M)$, the G-action permutes the factors in the tensor product. Since $T^n_R(M)$ is an invertible R-module, by Lemma 2.6.7, the group of R-automorphisms is isomorphic to $\mathrm{GL}_1(R) = R^*$. Hence G acts trivially on $T^n_R(M)$. We see that $\left(T^n_R(M) \otimes_R T^n_R(S)f\right)^G \cong T^n_R(M)$, and by θ, this is isomorphic to $\operatorname{Cor}^S_R(M_S)$. □

(05/27/2025) On p. 573, Theorem 14.1.13, Part (2) should be assumed false. The proof as given is based on Theorem 14.1.5, Part (3) which is false in general.

(05/20/2025) On p. 559, in Theorem 14.1.3, Part (2),
change: "T and S are algebras over Q" to: "T and R are algebras over Q"

(05/20/2025) On p. 562, in Theorem 14.1.5, Part (2),
change: "T and S are algebras over Q" to: "T and R are algebras over Q"

(05/11/2025) On p. 197, in the line immediately above Corollary 5.5.9,
change: "proves" to: "prove"

(05/09/2025) On p. 168, in the proof of Lemma 5.1.16,
change: “$\sum_{a_i} = 0$” to: “$\sum_i a_i = 0$”

(05/08/2025) On p. 390, in the proof of Theorem 10.3.5 (1) – (4),
change: “Proposition 4.4.1” to: “Theorem 4.4.1”

(04/23/2025) On p. 320, in the proof of Theorem 8.4.5 (2),
change: “Theorem 8.4.3” to: “Proposition 8.4.3”

(08/16/2024) On p. 475, in the proof of Corollary 12.7.4,
change: “Corollary 13.2.22” to: [DF04, Corollary 13.2.22]

(11/09/2023) I am grateful to Philippe Gille and M. Bruneaux for pointing this out to me. In the exact sequence of Proposition 10.4.9, the map ρ is not necessarily one-to-one. In the statement of Proposition 10.4.9, change the exact sequence of pointed sets to:

$$\check{\mathrm{H}}^1_{\mathrm{et}}(R, \mathbb{G}_m) \xrightarrow{\rho} \check{\mathrm{H}}^1_{\mathrm{et}}(R, \mathrm{GL}_n) \xrightarrow{\chi} \check{\mathrm{H}}^1_{\mathrm{et}}(R, \mathrm{PGL}_n) \xrightarrow{\partial} \check{\mathrm{H}}^2_{\mathrm{et}}(R, \mathbb{G}_m)$$

In the proof of Proposition 10.4.9, delete: “The proof that ρ is one-to-one is left to the reader.”

(07/31/2023) I am grateful to Erhard Neher for bringing to my attention the following problems that appear in Section 14.1 and for offering the suggestions for alternate proofs of Theorems 14.1.8 (3) and 14.1.9 (3) which are given below.

A goal on my to-do list is to correct Theorem 14.1.5 and rewrite the proofs of Theorems 14.1.8 (3) and 14.1.9 (3). Until then, Theorem 14.1.5 (3) should be assumed false.

It seems that the second part of [KO75, Theorem 3.2 (2)], which is part (3) of Theorem 14.1.5, is not correct. M. Ojanguren mentions this in his review of [Ver88] for Math Reviews.

If Theorem 14.1.5 (3) is not correct, this has an impact on the proof of Theorem 14.1.8 (3). However, we only need this for invertible modules, which has been proved in [Gro61, EGA II, (6.5.2.4)]. As pointed out in [KO75, Example 6.1], our norm of an invertible module agrees with that of [Gro61, EGA II]. (See the correction above, in the errata dated (05/28/2025).)

The same problem exists with the proof of Theorem 14.1.9 (3), which is also based on Theorem 14.1.5 (3). The result however is correct, as is shown in [Fer98, Remark 7.3.6], or in [Sal99, Corollary 8.2]. (See the correction above, in the errata dated (05/28/2025).)

(07/28/2023) I am grateful to Erhard Neher for pointing this out to me. On p. 581, line 4, change: Section 13.8.1 to: Section 7.8.1.

(07/28/2023) On p. 582, in Theorem 14.1.17, the cohomology groups in the commutative diagram should both be changed to H^1 groups. That is, change the commutative diagram to:

$$\begin{array}{ccc} \mathrm{B}(L'/S) & \xrightarrow{\alpha_5} & \mathrm{H}^1(G, \mathrm{Pic}(L')) \\ \Big\downarrow{\scriptstyle \mathrm{Cor}_R^S} & & \Big\downarrow{\scriptstyle \mathrm{Cor}_L^{L'}} \\ \mathrm{B}(L/R) & \xrightarrow{\alpha_5} & \mathrm{H}^1(G, \mathrm{Pic}(L)) \end{array}$$

(02/20/2023) On p. 261, in the penultimate sentence of the proof of Theorem 7.5.4, change: $m > 1$ to $[D^* : F^*] > 1$.

(10/11/2021) On p. 267, in Section 7.7: In the opening paragraph, change: “$R_0 \subseteq R$, where R_0 is a finitely generated $\mathbb{Z}$-algebra.” to: “$R_0 \subseteq R$, where R_0 is a finitely generated subring of R. If R is commutative, then R_0 can be taken to be a finitely generated $\mathbb{Z}$-algebra, hence can be taken to be noetherian.”

On p. 267, in Proposition 7.7.1 change: “Then there is a noetherian subring $R_0 \subseteq R$ (in fact R_0 can be taken to be a finitely generated $\mathbb{Z}$-algebra)” to: “Then there is a finitely generated subring $R_0 \subseteq R$ (if R is commutative, then R_0 can be taken to be a finitely generated $\mathbb{Z}$-algebra, hence can be taken to be noetherian)”

(09/13/2021) On p. 300, line 3, in the proof of (2) implies (1) of Theorem 8.1.24, change: Corollary 1.3.19 to Corollary 1.3.18.

(06/26/2021) On p. 371, in the proof of (3) implies (4) of Theorem 10.4.1, change: S_0 to B_0.

(07/03/2020) I wish to thank Nguyen Xuan Bach for pointing this out to me. On p. 269, in the proof of Proposition 7.7.2, there is a gap because the top rows of Diagrams (7.9) and (7.10) are not exact. To correct this error, replace the entire first paragraph of the proof with this:

The finitely generated projective R-module A is a direct summand of R^n, for some $n \geq 1$. Therefore, there is an idempotent a in $\mathrm{Hom}_R(R^n, R^n)$ and an exact sequence of R-modules

$$R^n \xrightarrow{a} R^n \xrightarrow{c} A \to 0. \tag{7.8}$$

Let $\{v_1, \dots, v_n\}$ be the standard basis for R^n. Without loss of generality, assume the image of v_1 under c is 1, the multiplicative identity of A. As in Proposition 7.7.1, the commutative diagram

$$\begin{array}{ccccccc} (R^n \otimes R^n) \oplus (R^n \otimes_R R^n) & \xrightarrow{a\otimes 1 + 1 \otimes a} & R^n \otimes_R R^n & \xrightarrow{c \otimes c} & A \otimes_R A & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mu} & & \\ R^n & \xrightarrow{a} & R^n & \xrightarrow{c} & A & \longrightarrow & 0 \end{array} \tag{7.9}$$

results from combining (7.8) with the multiplication map μ. The top row of (7.9) is exact, by Lemma 5.2.2. Let R_0 be the subring of R generated by the entries in the matrices of a, ψ, and ϕ with respect to the standard basis for R^n. Then the matrices descend to define R_0-module homomorphisms a_0, ψ_0, and ϕ_0. Define A_0 to be the cokernel of a_0. We get a commutative diagram

$$
\begin{array}{ccccccc}
(R_0^n \otimes_{R_0} R_0^n) \oplus (R_0^n \otimes_{R_0} R_0^n) & \xrightarrow{a_0\otimes 1+1\otimes a_0} & R_0^n \otimes_{R_0} R_0^n & \longrightarrow & A_0 \otimes_{R_0} A_0 & \longrightarrow & 0 \\
\downarrow \psi_0 & & \downarrow \phi_0 & & \downarrow \mu_0 & & \\
R_0^n & \xrightarrow{a_0} & R_0^n & \longrightarrow & A_0 & \longrightarrow & 0
\end{array}
\tag{7.10}
$$

where the top row is exact, by Lemma 5.2.2 and μ_0 is induced by the rest of the diagram. The proof of Proposition 7.7.1 shows that A_0 is a finitely generated projective R_0-module, $A = A_0 \otimes_{R_0} R$, and $A_0 \subseteq A$. Since (7.10) commutes, A_0 is an R_0-subalgebra of A. This proves (1) and (2).

(03/13/2018) On p. 256, in the statement of Theorem 7.4.3,
change: $\operatorname{Rank}_{R_\mathfrak{p}}(A_\mathfrak{p})$ to: $\operatorname{Rank}_{R_\mathfrak{p}}(B_\mathfrak{p})$.

(11/4/2017) On p. 62, in Lemma 2.2.7 part (2), add a period after the displayed equation.

(11/4/2017) On p. 19, in Section 1.2.3, line 3,
change: . . . purpose of this section is to proof . . .
to: . . . purpose of this section is to prove . . . .

(9/26/2017) On p. 614, in Exercise 14.3.19,
change: $R = k[x, y](xy - 1)$ to: $R = k[x, y]/(xy - 1)$
change: $R = k[x, y](x^2y - 1)$ to: $R = k[x, y]/(x^2y - 1)$
change: $R = k[x, y](y^2 - x^3 + x)$ to: $R = k[x, y]/(y^2 - x^3 + x)$
change: $R = k[x, z](z - x^3 + xz^2)$ to: $R = k[x, z]/(z - x^3 + xz^2)$

(7/5/2017) On p. 602, in Section 14.3.3 in the third paragraph (the paragraph that defines the ring R), between sentences one and two, add: Assume $p(x)$ is not a square.

(7/5/2017) On p. 595, in the proof of Theorem 14.2.12, in the last paragraph of Step 1, change: ∂ to: ∂_1.

References

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