PROOF OF EXAMPLE 9.2.4 (3) (AN OPEN IMMERSION IS ÉTALE)

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The purpose of this note is to prove that if $f : R \to S$ is a homomorphism of commutative rings and $f^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is an open immersion, then S is an R-algebra of finite presentation (see Corollary 10 below). This theorem is referenced without proof in [1, Example 9.2.4 (3)]. The corollary and its proof presented here are based on [3, Proposition I.4.6, p. 328]. The proof below utilizes only theorems proven in [2] and [1].

Throughout, all rings are commutative. Let M be an R-module and α a nonzero element of R. Denote by M_α the localization of M with respect to the multiplicative set $\{1, \alpha, \alpha^2, \alpha^3, \dots\}$.

Definition 1. Let R be a commutative ring and M an R-module. We say M is *locally finitely generated*, if there exist elements $\alpha_1, \dots, \alpha_n$ in R such that $R = R\alpha_1 + \cdots + R\alpha_n$ and for each i, M_{α_i} is a finitely generated R_{α_i}-module.

Lemma 2. *Let M be an R-module. If M is locally finitely generated, then M is finitely generated.*

Proof. Let $\alpha_1, \dots, \alpha_n$ in R such that $R = R\alpha_1 + \cdots + R\alpha_n$ and for each i, M_{α_i} is a finitely generated R_{α_i}-module. For each i, let $\{x_{ij} \mid 1 \le j \le n_i\}$ be a subset of M which is a generating set for M_{α_i} as an R_{α_i}-module. The reader should verify that $\bigcup_i \{x_{ij} \mid 1 \le j \le n_i\}$ is a generating set for the R-module M. □

Definition 3. Let S be a commutative R-algebra with structure homomorphism $\theta : R \to S$. We say S is an *R-algebra of finite presentation*, if there is a polynomial ring in $n \ge 1$ variables over R and a surjective R-algebra homomorphism $\phi : R[x_1, \dots, x_n] \to S$ such that the kernel of ϕ is a finitely generated ideal. The R-algebra S is said to be *locally of finite presentation*, if there exist elements $\beta_1, \dots, \beta_n$ in S such that $S = S\beta_1 + \cdots + S\beta_n$ and for each i, S_{β_i} is an R-algebra of finite presentation. We also say θ is *locally of finite presentation.*

Example 4. Let α be a non-zero element of R. Define $\phi : R[x] \to R_\alpha$ by $\phi(x) = 1/\alpha$. The kernel of ϕ is the principal ideal generated by $x\alpha - 1$. This shows R_α is an R-algebra of finite presentation.

Lemma 5.
1. *(Change of Base) Let $f : R \to T$ be a homomorphism of commutative rings such that T is an R-algebra of finite presentation. If $g : R \to S$ is any homomorphism of commutative rings, then $f \otimes 1 : S \to T \otimes_R S$ makes $T \otimes_R S$ into an S-algebra of finite presentation.*
2. *(Finitely Presented over Finitely Presented is Finitely Presented) If S is a finitely presented R-algebra and T is a finitely presented S-algebra, then T is a finitely presented R-algebra.*

Proof. (1): There exists an onto homomorphism $\phi : R[x_1, \dots, x_n] \to T$ such that $\ker \phi$ is a finitely generated ideal. Then $\phi \otimes 1 : S[x_1, \dots, x_n] \to T \otimes_R S$ is onto. The kernel of $\phi \otimes 1$ is generated by the image of $\ker \phi \otimes_R S$, hence is a finitely generated ideal.

(2): There exist homomorphisms $\phi : R[x_1, \dots, x_m] \to S$ and $\psi : S[y_1, \dots, y_n] \to T$ such that ϕ and ψ are both onto, and $\ker \phi$ and $\ker \psi$ are both finitely generated ideals. Define $\chi : R[x_1, \dots, x_m][y_1, \dots, y_n] \to T$ by $\chi(x_i) = \psi\phi(x_i)$ for each i and $\chi(y_i) = \psi(y_j)$ for each j. Then χ is onto and the kernel of χ is generated by $\ker \phi + \ker \psi$, hence is finitely generated. □

Proposition 6. *Let $f : R \to S$ be a homomorphism of commutative rings. The following are true.*

(1) *If the kernel of f is a finitely generated ideal and f is onto, then S is an R-algebra of finite presentation.*
(2) *If f is one-to-one and S is a finitely generated R-module, then S is an R-algebra of finite presentation.*
(3) *If the kernel of f is a finitely generated ideal and S is a finitely generated R-module, then S is an R-algebra of finite presentation.*

Proof. The proof is left to the reader as an exercise. □

Proposition 7. (1) *If $f : R \to S$ is a local isomorphism, then f is locally of finite presentation.*
(2) *(Composition) If $f : R \to S$ and $g : S \to T$ are both locally of finite presentation, then $g \circ f : R \to T$ is locally of finite presentation.*
(3) *(Change of Base) Let $f : R \to T$ and $g : R \to S$. If f is locally of finite presentation, then $f \otimes 1 : S \to T \otimes_R S$ makes $T \otimes_R S$ into an S-algebra that is locally of finite presentation.*
(4) *(Tensor Product) Let $f : S \to S'$ and $g : T \to T'$ be R-algebra homomorphisms. If f and g are both locally of finite presentation, then $f \otimes g : S \otimes_R T \to S' \otimes_R T'$ is locally of finite presentation.*
(5) *(Descent) If $f : R \to S$ is locally of finite type and $g : S \to T$, and $g \circ f : R \to T$ is locally of finite presentation, then g is locally of finite presentation.*

Proof. (1): For every $\mathfrak{q} \in \operatorname{Spec} S$ there exists $\alpha \in R$ such that $f(\alpha) \in S - \mathfrak{q}$ and $R_\alpha \to S_\alpha$ is an isomorphism. By Example 4, S_α is an R-algebra of finite presentation. By [2, Exercise 3.3.29], S is locally of finite presentation.

(2): Let $\mathfrak{r} \in \operatorname{Spec} T$ be arbitrary. There exists $\beta \in T - \mathfrak{r}$ such that $g : S \to T_\beta$ is of finite presentation. Let $\mathfrak{q} = \mathfrak{r} \cap S$. There exists $\alpha \in S - \mathfrak{q}$ such that $f : R \to S_\alpha$ is of finite presentation. By Lemma 5 (1), $g : S_\alpha \to T_{\beta\alpha}$ is of finite presentation. By Lemma 5 (2), $g \circ f : R \to S_\alpha \to T_{\beta\alpha}$ is of finite presentation. Because $\operatorname{Spec} S$ is compact ([2, Exercise 3.3.29]), we see that T is locally of finite presentation as an R-algebra.

(3): Start with elements $\beta_1, \dots, \beta_n$ in T such that $T = T\beta_1 + \cdots + T\beta_n$ and for each i, T_{β_i} is an R-algebra of finite presentation. Then $\beta_1 \otimes 1, \dots, \beta_n \otimes 1$ generate the unit ideal in $T \otimes_R S$. By Lemma 5 (1), $T_{\beta_i} \otimes_R S$ is an S-algebra of finite presentation, for each i.

(4): By (3), $f \otimes 1 : S \otimes_R T \to S' \otimes_R T$ and $1 \otimes g : S' \otimes_R T \to S' \otimes_R T'$ are both locally of finite presentation. By (2), $f \otimes g$ is locally of finite presentation.

(5): First note that $S \otimes_{S \otimes_R S} (T \otimes_R S)$ is isomorphic to T. The diagram

$$\begin{array}{ccccc} & & T & & \\ & \overset{\rho}{\nearrow} & & \overset{g}{\nwarrow} & \\ T \otimes_R S & & & & S \\ & \underset{g\otimes 1}{\nwarrow} & & \underset{\mu}{\nearrow} & \\ & & S \otimes_R S & & \end{array}$$

commutes, where $\rho(t \otimes s) = ts$. By [2, Proposition 3.5.14], f makes S into a finitely generated R-algebra. By [2, Lemma 10.1.6], the multiplication homomorphism $\mu : S \otimes_R S \to S$ makes S into a finitely presented $S \otimes_R S$-module. By Proposition 6, S is a finitely presented $S \otimes_R S$-algebra. By (3), ρ makes T into a $T \otimes_R S$-algebra of finite presentation. By (3) again, $(g \circ f) \otimes 1 : S \to T \otimes_R S$ is locally of finite presentation. The diagram

$$\begin{array}{ccccc} S & & \xrightarrow{\quad g \quad} & & T \\ & \underset{(g\circ f)\otimes 1}{\searrow} & & \underset{\rho}{\nearrow} & \\ & & T \otimes_R S & & \end{array}$$

commutes, where $\rho(t \otimes s) = ts$. By (2), g is locally of finite presentation. □

Lemma 8. *Let I be a proper ideal in R and α a nonzero element of R such that R_α / I_α is an R-algebra of finite presentation. Then I_α is a finitely generated ideal in R_α.*

Proof. There exists an onto R-algebra homomorphism $\phi : R[x_1, \dots, x_n] \to R_\alpha / I_\alpha$ such that $\ker \phi$ is a finitely generated ideal. Tensor with $(\cdot) \otimes_R R_\alpha$ to get the exact sequence

$$0 \to \ker \phi \otimes_R R_\alpha \to R_\alpha[x_1, \dots, x_n] \xrightarrow{\tau} R_\alpha / I_\alpha \to 0$$

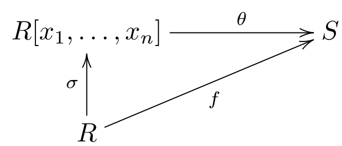
where $\tau = \phi \otimes 1$. Therefore, $\ker \tau = \ker \phi \otimes_R R_\alpha$ is a finitely generated ideal. Let $\eta : R_\alpha \to R_\alpha / I_\alpha$ be the natural map. Since τ is an R_α-algebra homomorphism, there exists a lifting $\psi : R_\alpha[x_1, \dots, x_n] \to R_\alpha$ such that $\eta\psi(x_i) = \tau(x_i)$ for each i. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \tau & \longrightarrow & R_\alpha[x_1, \dots, x_n] & \xrightarrow{\tau} & R_\alpha / I_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle u} & & \\ 0 & \longrightarrow & I_\alpha & \longrightarrow & R_\alpha & \xrightarrow{\eta} & R_\alpha / I_\alpha & \longrightarrow & 0 \end{array}$$

commutes and the rows are exact sequences. Since ψ is onto and u is the identity map, the Snake Lemma ([2, Theorem 2.3.2]) implies $\psi(\ker \tau) = I_\alpha$. This proves I_α is a finitely generated ideal in R_α. □

Proposition 9. *If $f : R \to S$ is locally of finite presentation, then f is of finite presentation.*

Proof. There is an onto R-algebra homomorphism $\theta : R[x_1, \ldots, x_n] \to S$. Therefore f factors through θ and the diagram



commutes where σ is the natural map. Let J denote the kernel of θ. By Proposition 7 (5), θ is locally of finite presentation. By Lemma 8, J is locally a finitely generated ideal in $R[x_1, \ldots, x_n]$. By Lemma 2, J is a finitely generated ideal. Hence, S is an R-algebra of finite presentation. $\square$

Corollary 10. *Let $f : R \to S$ be a homomorphism of commutative rings. If $f^\sharp : \operatorname{Spec} S \to \operatorname{Spec} R$ is an open immersion, then S is an R-algebra of finite presentation.*

Proof. By Proposition 7 (1), $f : R \to S$ is locally of finite presentation. By Proposition 9, $f : R \to S$ is of finite presentation. $\square$

References

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[2] ———, *Commutative algebra*, Open Math Notes, American Mathematical Society, Reference # OMN:202409.111443, 2024, the latest version is available at the URL: `https://tim4datfau.github.io/Timothy-Ford-at-FAU/preprints/CA.pdf`.

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