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DIVISION ALGEBRAS OVER HENSELIAN SURFACES

TIMOTHY J. FORD AND DAVID J. SALTMAN

Department of Mathematics
Florida Atlantic University
Boca Raton, Florida 33431

Department of Mathematics
The University of Texas
Austin, Texas 78712

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Abstract. Let Y be a normal surface defined over an algebraically closed field of characteristic 0. Let R' be the local ring of R at a closed point and R the completion or henselization of R'. Set K to be the field of fractions of R. If D/K is a finite dimensional division algebra with center K we show D is a cyclic algebra. That is, $D \cong (\alpha, \beta)_{n,K}$ When R has a rational singularity we describe such an α, β in terms of the ramification of D.

0. Introduction

There is a rather small class of fields K for which there is good information about all division algebras D finite dimensional over their center K. Prominent among such fields is the class of global fields. Let K be a global field, and write D/K to mean D is a division algebra finite dimensional over its center K. Then D defines an element $[D]$ in the Brauer group $\mathrm{Br}(K)$. The exponent of D is the order of $[D]$ and the degree of D is the square root of the dimension $[D : K]$. As part of the classical theory of division algebras over global fields (e.g., [Re], p.280) one knows that D has exponent equal to its degree. furthermore, any such D is a cyclic algebra. One has a description of the splitting fields of D in terms of the so called Hasse invariants of D.

The goal of this paper is to present another class of fields K and results about all division algebras D/K. To this end, let Y be an algebraic surface defined over an algebraically closed field F of characteristic 0. If $P \in Y$ is a closed normal point, let R' be the local ring of Y at P. Set R to be either the (strict) henselization or completion of R' with respect to the maximal ideal. Set $K = q(R)$ to be the field of fractions of the domain R. In [A], Artin showed that every division algebra D/K has exponent equal to its degree. Using some of Artin's basic results, we give further results about such D/K. We reprove Artin's result, and in addition show that all such D/K are cyclic algebras. To prove

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such a result about D/K, one first of all notes that (just as in [A]) by Artin approximation ([A2]) we can restrict to the case that R is the henselization of R'. Second, we note that it suffices by e.g., [Al], p.60 or [Re], p.261 to find a cyclic field extension L/K such that L splits D and has degree equal to the exponent of D. This we do, and along the way we give a description of all the Galois splitting fields of D in terms of the ramification locus of D on certain blow ups of a desingularization of Spec(R). A key role in our argument will be played by surfaces with rational singularities.

Since K contains all roots of one, any cyclic algebra of degree n with center K is a "symbol algebra" $(a,b)_{n,K}$. Fix a primitive n root of 1, ρ. Recall that $(a,b)_{n,K}$ is generated over K by α, β subject to the relations $\alpha^n = a$, $\beta^n = b$, and $\alpha\beta = \rho\beta\alpha$. Given the ramification data for a D/K, the method of proof for the results mentioned so far give a description of an element "a" in K such that $D \cong (a,b)_{n,K}$, but no description is given of the "b". In the last section, we give a different proof that D/K is cyclic in the case R has a rational singularity, with the additional virtue of describing both "a" and "b".

Let us recall some basic facts and prove some preliminary results. Let K be an arbitrary field and $v : K^* \to \mathbb{Z}$ a discrete valuation on K. Denote by T the associated valuation ring. There is an exact sequence ([AB], p.289):

$$0 \to \mathrm{Br}(T) \to \mathrm{Br}(K) \xrightarrow{\chi^T} \mathrm{Hom}(G_T, \mathbb{Q}/\mathbb{Z}) \to 0 \tag{1}$$

where G_T is the absolute Galois group of the residue field, k, of T; Hom refers to continuous homomorphisms; and $\mathbb{Q}/\mathbb{Z}$ has the discrete topology. We call χ^T the character map. If $f \in \mathrm{Hom}(G_T, \mathbb{Q}/\mathbb{Z})$ then f has finite and hence cyclic image. The kernel of f then defines a cyclic Galois extension L/k and we say L/k is the cyclic extension defined by f. $\mathrm{Hom}(G_T, \mathbb{Q}/\mathbb{Z})$ is also the étale cohomology group $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ and we will use both expressions interchangeably.

If X is a two dimensional integral normal scheme then any irreducible curve $C \subseteq X$ defines a discrete valuation on the function field K of X. Thus for each such C there is an associated character map $\chi^C : \mathrm{Br}(K) \to \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$. If $[D] \in \mathrm{Br}(K)$, it is very easy to see that $\chi^C([D]) = 0$ for all but finitely many C. The C for which $\chi^C([D]) \neq 0$ are called the ramification curves of D, and the set of ramification curves and the associated $\chi^C([D]) \in \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$ is called the ramification data of D.

To describe how this ramification data "fits together", we make the following definitions. Fix an isomorphism of $\mathbb{Q}/\mathbb{Z}$ with the group of roots of 1. More precisely, for all n, choose a primitive n root of 1, $\rho(n) \in F$, such that $\rho(nm)^m = \rho(n)$. Let C be a curve over F and $C' \to C$ the normalization of C. Denote by k the function field of C. For a point $P \in C$ let $P_1, \dots, P_m \in C'$ be the points lying over P. For any $f \in \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$, define the "ramification" $r_i(f) \in \mathbb{Q}/\mathbb{Z}$ as follows. Let k_i be the completion of k with respect to the valuation defined by P_i, and let M_i be the algebraic closure of k_i. M_i is the union of fields $k_i(\pi_n)$ such that $(\pi_n)^n \in k_i$ is a prime element. There is a unique generator $\sigma_i \in \mathrm{Gal}(M_i/k_i)$ such that $\sigma_i(\pi_n) = \rho(n)\pi_n$ for all n. The map f restricts to an $f_i : \mathrm{Gal}(M_i/k_i) \to \mathbb{Q}/\mathbb{Z}$ and $r_i(f) = f_i(\sigma_i)$. If f defines L/k and L_i is the completion of L with respect to a point over P_i, then the order of $r_i(f)$ in $\mathbb{Q}/\mathbb{Z}$ is the degree of L_i/k_i which is also the ramification degree of L/k at P_i. Finally, define $r_{P,C}(f)$ to be the sum of the $r_i(f)$.

Let X be an irreducible, regular, two dimensional scheme which is the direct limit of such schemes of finite type over F. Set K to be the function field of X. The map r defined above is used in describing a necessary restriction on the ramification data of a division algebra D/K.

Proposition 0.1. *The composition*

$$\mathrm{Br}(K) \xrightarrow{\chi^X} \bigoplus_{C \subseteq X} \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{P \in X} \mathbb{Q}/\mathbb{Z} \tag{2}$$

is zero, where:

1) The first direct sum is over all irreducible curves $C \subseteq X$
2) The second direct sum is over all closed points
3) The map χ^X is the sum of all the character maps χ^C
4) The map r is the sum of all

$$r_P : \bigoplus_{C \subseteq X} \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

and the r_P themselves are defined to be $r_{P,C}$ on any $\mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$ where C contains P and 0 otherwise.

The proof of the above result is in [AM], but we do not assume $\mathrm{H}^3(X, \mathbb{Q}/\mathbb{Z}) = (0)$ and so cannot conclude (2) is exact.

A very important consequence of 0.1 is:

Corollary 0.2. *Let X be as above and $C \subseteq X$ a finite tree of complete nonsingular rational curves. Assume $[D] \in \mathrm{Br}(K)$ satisfies $\chi^{X-C}([D]) = 0$. Then $\chi^X([D]) = 0$. In other words, if $[D]$ is unramified on the complement of C, then $\chi^X[D] = 0$.*

Proof. Note first of all that any cover of $\mathbb{P}^1$ ramifies for at least two points. That is, $\mathbb{A}^1$ is simply connected (e.g., [M] p.42). This can be seen directly for cyclic covers, the only case we need, by observing the following. If $f \in F[x]$, $K = F(x)$, and $L = K(f^{1/n})$ is unramified over $F[x]$, then each zero of f has order multiple of n and hence f is an n-th power.

Since C is a finite tree, there is a curve $\mathbb{P}^1 \cong L \subseteq C$ such that $L \cap \overline{(C - L)}$ is one point. Since $\chi^L([D])$ ramifies at least two points, 0.1 implies $\chi^L([D]) = 0$. If $C' = \overline{(C - L)}$, then $\chi^{X-C'}([D]) = 0$. By induction on the number of components of C, the proof is done. □

Let us note that in the applications of 0.2 in this paper we will know that χ^X is injective so 0.2 will imply $D = K$ (or $[D] = 1$).

As a final remark in this section, let R be a two dimensional local henselian domain, F its residue field, and $P \subseteq R$ a prime with R/P of dimension 1. Let k be the field of fractions of R/P, and let G_k be the absolute Galois group of k. Then R/P is henselian ([R], p.8), and so is the normalization R' of R/P in its field of fractions ([R], p.7). By [R], p.7, R' is local and hence is a henselian discrete valuation ring. As all field extensions L/k are totally and tamely ramified it follows that $r : \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism. It is useful to think of R/P as a curve with one point and every cover must ramify at that point.

1. Splitting fields

Let us recall our basic situation. R is the henselization of a closed point on a normal algebraic surface over an algebraically closed field F of characteristic 0. If $K = q(R)$ is the field of fractions of R, we will study the splitting fields of elements $\alpha \in \mathrm{Br}(K)$. In particular, we will show that if α has exponent n, then α has a cyclic splitting field of degree n. In other language, if D/K is a division algebra with center K and of exponent n, then D is a cyclic algebra of degree n.

Let $L \supseteq K$ be a finite field extension and S the integral closure of R in L. Let $Y \to \mathrm{Spec}(S)$ be a resolution of the singularities of $\mathrm{Spec}(S)$. Since S itself is the henselization of the closed point of a surface over F, Artin showed that the character map

$$\mathrm{Br}(L) \to \bigoplus_{C \subseteq Y} \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z}) \tag{3}$$

is an embedding, where the direct sum is over all irreducible curves in Y. In other words, the splitting of $\alpha \in \mathrm{Br}(K)$ by L reduces to showing that α maps to 0 in each $\mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$.

For any irreducible curve $C \subseteq Y$, let v_C be the associated discrete valuation. Then v_C restricts to a discrete valuation w_C on K. The valuation w_C has a residue field with absolute Galois group we denote by G'_C. Let $e = e(v_C/w_C)$ be the ramification degree. Using the definitions one can easily check that there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(L) & \longrightarrow & \mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow e \\ \mathrm{Br}(K) & \longrightarrow & \mathrm{Hom}(G'_C, \mathbb{Q}/\mathbb{Z}) \end{array} \tag{4}$$

where $\mathrm{Br}(K) \to \mathrm{Br}(L)$ is the restriction map and the map "e" is the integer e times the canonical map induced by $G_C \subseteq G'_C$. To show L splits α it is enough to show L "splits" the image of α in $\mathrm{Hom}(G'_C, \mathbb{Q}/\mathbb{Z})$ for all w_C that arise. That is, it is enough to show that α maps to 0 in $\mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$ for all C that arise. When L/K is Galois, all extensions of w_C are conjugate under the Galois group. Thus α maps to 0 in $\mathrm{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$ for one extension if and only if α maps to 0 for all extensions. When this happens, we say L splits α at C. Thus L splits α if and only if L splits α at all possible C.

The difficulty here is that not knowing L, it is not clear which w_C must be considered. If $X \to \mathrm{Spec}(R)$ is a resolution of singularities, w_C may not correspond to a curve on X, but to one on a blow up of X. So the difficulty is to determine how to blow up X so that all w_C appear.

Given X, and the ramification locus of L, one could try to describe a blowing up of $X' \to X$ such that the normalization, Y, of X' in L is nonsingular. In particular, any curve in Y would then lie over a curve in X'. However, blowing up to achieve nonsingularity is unnecessary. Following a hint in Artin ([A]), we weaken the requirement on Y and show that we only need that Y have rational singularities. We can then give a simple description of the property X' requires so that its normalization Y has rational singularities.

To recall the definition, let R'' be a local normal two dimensional F algebra, and $\eta : X'' \to \mathrm{Spec}(R'')$ a resolution of singularities. Then R'' has a rational singularity if $\mathrm{H}^1(X'', \mathcal{O}_{X''}) = 0$. A two dimensional scheme Y has rational singularities if each local ring $\mathcal{O}_y$ has one, for y a closed point. As it turns out, we will show our varieties have rational singularities using the following theorem of Boutot ([B]) (true in any dimension). Let G be a linear reductive group over F and assume G acts rationally on a commutative F algebra, A, with rational singularities. Then the fixed ring A^G has rational singularities.

We begin with a well known lemma, leading up to 1.2.

Lemma 1.1. *Let R, M be a regular local dimension two F algebra with $R/M = F$.*

Assume $f, g \in M$ is a system of parameters. Set $S = R[y]/(y^n - f)$, and let $x \in S$ be the image of y. Then S is a domain, a regular local ring, and x, g is a system of parameters for S. In particular, S is the integral closure of R in $q(S)$.

Proof. Let $N = M + xR + \cdots + x^{n-1}R \subseteq S$. Then N is an ideal and $S/N = F$. Hence N is maximal. Any other maximal ideal containing M also contains x and so is N. Thus S is local. As R is a unique factorization domain, $y^n - f$ is irreducible and S is a domain. Finally, x and g clearly generate N. □

As stated above, our goal is to give conditions on the ramification of the cover that force the cover to have rational singularities. Let R be a regular local ring of dimension two. Assume L is a finite separable field extension of $q(R) = K$ and that S is the integral closure of R in L. S is a reflexive R module because the double dual S^{**} contains S, is naturally embedded in L, is closed under multiplication (e.g., argue as in [OS], p.64), and is finite over R. By e.g., [OS], p.71, S is then projective as an R module. By the purity of branch loci (e.g., [M], p.24), the different $\delta_{S/R} \subseteq S$ has pure height one. Define the ramification locus $\mathrm{ram}(S/R)$ to be the set of height one primes $q \subseteq R$ such that $q = p \cap R$ for p a prime in S with p minimal over $\delta_{S/R}$. Thus S_p/R_q, is unramified, and hence étale, if and only if $q \notin \mathrm{ram}(S/R)$. In other terms, S/R is étale if and only if S_p/R_q, is unramified for all $q \subseteq R$ of height one and all primes $p \subseteq S$ lying over q. We say $\mathrm{ram}(S/R)$ has normal crossings if $\mathrm{ram}(S/R) = \{(f), (g)\}$ where f, g are a system of parameters for R. We will now state the needed result, whose proof will follow Lemma 1.3.

Theorem 1.2. *Let R be a regular, dimension two local ring, L a finite Galois extension of $K = q(R)$, and S the integral closure of R in L. If the ramification locus of S/R has normal crossings, then S has rational singularities.*

Continuing with the above set up, assume L/K is Galois. Then the set of primes $p \subseteq S$ lying over a given $q \subseteq R$ are all conjugate. In particular, there is a well defined ramification degree $e_q(L/K)$ being the ramification degree of S_p/R_q, for any p lying over q. Thus $q \in \mathrm{ram}(S/R)$ if and only if $e_q(L/K) > 1$. The key lemma used to prove 1.2 can now be stated.

Lemma 1.3. *Suppose S/R are as in 1.2 and $\mathrm{ram}(S/R) = \{(f), (g)\}$ has normal crossings. Assume n is a multiple $e_q(S/R)$ for each $q \in \mathrm{ram}(S/R)$. Set $K' = K(f^{1/n}, g^{1/n})$ to be the field extension of K and R' the integral closure of R in K'. Set L' to be the compositum of K' and L and S' the integral closure of R in L'. Then S'/R' is étale.*

Proof. Let x, y be such that $x^n = f$ and $y^n = g$. Then R' is a regular local ring with $\{x, y\}$ as a system of parameters by 1.1. By the above remarks, it suffices to show $e_q(L'/K') = 1$ for all height one primes $q' \subseteq R'$. If q' does not lie over (f) or (g), this is clear. It suffices then to assume (by symmetry) that $q = (f) = q' \cap R$. Clearly $q = xR'$. Set $K'' = K(f^{1/n})$, $R'' = R' \cap L''$, $q'' = q' \cap R''$, L'' to be the compositum of L and K'', and $S'' = S' \cap L''$. Choose $p' \subseteq S'$ a prime lying over q' and set $p'' = p' \cap S''$ and $p = p' \cap S$. We have the following diagram all the arrows of which denote inclusions.

$$\begin{array}{ccc} R'_{q'} & \longrightarrow & S'_{p'} \\ \uparrow & & \uparrow \\ R''_{q''} & \longrightarrow & S''_{p''} \\ \uparrow & & \uparrow \\ R_q & \longrightarrow & S_p \end{array}$$

As K'/K'' and L'/L'' are defined by adjoining an n-th root of g, $e_{q''}(K'/K'') = e_{p''}(L'/L'') = 1$. Thus it suffices to show $e_{q''}(L''/K'') = 1$.

Let M, M'', N, N'' be the completions of K, K'', L, L'' with respect to the valuations defined by q, q'', p, p'' respectively. These complete fields have canonical valuations we need not specify explicitly. We have:

$$\begin{array}{ccc} M'' & \subseteq & N'' \\ \uparrow & & \uparrow \\ M & \subseteq & N \end{array}$$

If we set $e = e(N/M) = e_q(L/K)$, then recall that e is a divisor of n. By e.g., [CF], p.27, there is an intermediate field $M \subseteq M_1 \subseteq N$ such that M_1/M is unramified and N/M_1 is totally and tamely ramified of degree e. By e.g., [CF], p.32, $N = M_1\big((uf)^{1/e}\big)$ for u a unit of M_1. Since $N'' = N(f^{1/n})$, $e(N''/N) = n/e$. Thus $e_{q''}(L''/K'') = e(N''/M'') = e(N''/N)e(N/M)/e(M''/M) = (n/e)e/n = 1$. $\square$

Now to give the proof of 1.2 is an easy matter. With the notation as in 1.3, S' is a regular ring, and L'/K is Galois with group say G. Let $H \subseteq G$ be the subgroup fixing L. Then S is the fixed ring S'^H and the result follows from Boutot's theorem [B].

The next result makes good the claim that rational singularities are "good enough".

Lemma 1.4. *Let R be as above and $Y \to \operatorname{Spec}(R)$ a birational proper map such that Y has rational singularities. If $K = q(R)$, then the character map:*

$$\operatorname{Br}(K) \to \bigoplus_{C \subseteq Y} \operatorname{Hom}(G_C, \mathbb{Q}/\mathbb{Z})$$

is injective, the direct sum being over all irreducible curves on Y.

Proof. Let $Y' \to Y$ be a resolution of singularities of Y (and hence Spec(R)). By [L], p.204 proof of 4.1, Y' can be constructed by blowings up alone. In particular, the exceptional divisors of Y' not from Y form a tree. Suppose $\alpha \in \text{Br}(K)$ is in the kernel of the map above. By (3), α must ramify along these exceptional curves. But now the result follows from 0.2. □

As a consequence of the above we have:

Proposition 1.5. *Let R be as above and $X \to \text{Spec}(R)$ a resolution of the singularities of* Spec(R). *Assume L/K is Galois and Y the integral closure of K in L. Assume the ramification of Y/X has only normal crossings. Then L is a splitting field of α if and only if L splits α on any curve of X.*

The above result is a concrete description of the splitting fields of any element $\alpha \in \text{Br}(K)$. Given L/K, we blow up X until the ramification of α and L/K has normal crossings. If $Z \subseteq X$ is the ramification of L/K and α on X, it suffices to construct a blow up $X' \to X$ such that the inverse image of Z has normal crossings and this is a standard construction (e.g., [H], p.391). Given X', then we "test" L by looking at L restricted to any of the (finitely many) curves along which α ramifies on X' and check whether L splits the ramification by using (4). As an application, we show that if α has exponent n, α has a cyclic splitting field of degree n.

We have to be a bit more specific about the construction of R and our blow ups. Let R be the henselization of R', where R' is the localization at a closed point of a normal dimension two projective variety Y of finite type over F. Assume $\alpha \in \text{Br}(K)$. As R is the direct limit of étale covers of R', we may assume that α is in the image of $\text{Br}(K')$ where $K' = q(R')$. Let $X' \to Y$ be a resolution of singularities. Let $C_1, \ldots, C_r$ be the curves on X' along which α ramifies. Construct a blow up $X'' \to X'$ such that if $Z \subseteq X'$ is the exceptional divisor, Z union the proper transforms of the C_i's have normal crossings. Rename things so that Z union these C_i's have $\{E_1, \ldots, E_s\}$ as underlying curves. Let $E = -E_1 - \cdots - E_s$. According to [H], p.358: proof of 1.1, $E = H_1 - H_2$ where the H_i are very ample divisors. By[H], p.358 Lemma 1.2 (essentially Bertini's theorem) there are nonsingular curves D_1, D_2 such that D_i is in the linear system $|H_i|$ and $D_1 \cup D_2 \cup E_1 \cup \cdots \cup E_s$ has normal crossings. Hence there is an $f \in F(Y) = K'$ with $(f) = E_1 + \cdots + E_s + D_1 - D_2$.

Set $X = X' \times_Y R$, so $X \to \operatorname{Spec}(R)$ is a resolution of singularities and X/X' is a direct limit of étale extensions. It follows that the divisor $(f) = \sum \pm E'_i$ on X still has normal crossings and the E'_i are all distinct. Of course, the E'_i correspond to a subset of the E_i's, D_1 and D_2. In addition, the curves on X along which $\alpha \in \operatorname{Br}(K)$ ramifies are among the E'_i. Let n be the exponent of α and set $L = K(f^{1/n})$. If v_i is the valuation defined by E'_i, then $v_i(f) = 1$. Hence if e_i is the ramification degree of L/K at v_i, $e_i = n$. It follows from (4) that L splits α along every curve of X. The ramification of L/K is just (f) and so has normal crossings. Thus by 1.5, L splits α. We have proved:

Theorem 1.6. *Let R be the henselization of a closed normal point on a surface of finite type over an algebraically closed field of characteristic zero. Let K be the field of fractions of R and $\alpha \in \operatorname{Br}(K)$ an element of exponent n. Then $\alpha = [D]$ where D is a cyclic division algebra of degree n.*

That is, all division algebras over K are cyclic with degree equal to their exponent.

2. An explicit construction

As in Section 1, R is the henselization at a closed point of a normal algebraic surface over the algebraically closed field k of characteristic zero. Moreover, in this section we assume R has a rational singularity. Let K be the quotient field of R and A a central division algebra over K of exponent n in $\operatorname{Br}(K)$. Fix a primitive n-th root of unity ρ. Throughout, symbol algebras $(\alpha, \beta)_n$, will be formed over K with respect to ρ. By the results of Section 1, A has a cyclic splitting field of degree n, hence is a symbol algebra $(\alpha, \beta)_n$ for some $\alpha, \beta \in K^*$ ([Re], Theorem 30.3). The purpose of this section is to provide another proof of this result by explicitly exhibiting α and β, in the case where R has a rational singularity. The main result of this section is

Theorem 2.1. *In the above context, A is a symbol algebra $(\alpha, \beta)_n$. In particular,*

$$\operatorname{index}(A) = \operatorname{exponent}(A) .$$

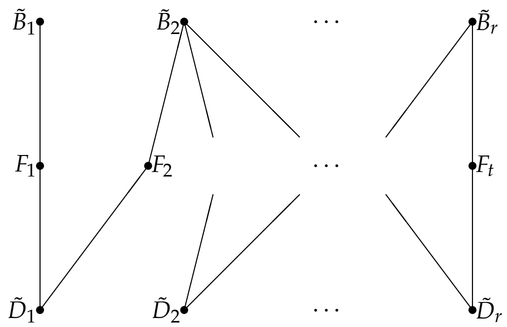
The proof of Theorem 2.1 takes up the rest of this section and is divided into a sequence of lemmas. First we establish some notation. As in Section 1 we fix an identification of the group of roots of 1 sheaf μ with $\mathbb{Q}/\mathbb{Z}$. Let $\pi : X \to \operatorname{Spec} R$ be a resolution of the singularities of R. From [L], proof of 4.1, we know that we can pick π to be a product of "blow up" maps. In particular, the closed fiber of π is a tree of smooth rational curves. We also know from [L], Theorem 17.4, that the divisor class group of R, $\operatorname{Cl}(R)$, is finite. Assume B is a reduced curve on X containing the underlying curve of the closed fiber of π. Let $B = B_1 \cup \cdots \cup B_r$

where the B_i's are prime divisors on X. Assume also that B contains the ramification divisor of the algebra A and that B is a divisor with normal crossings ([A], Section 1). Denote by σB the singular locus of B, $\sigma B = \{B_i \cap B_j | i \neq j\}$. Let $|\,|$ denote cardinality of sets. If $|\sigma B| = s$, let $\pi_1 : X_1 \to X$ be the blowing-up of the s points in σB. Let $D_1, \dots, D_s$ be the new exceptional lines and write $C = \pi_1^{-1}(B) = D_1 \cup \cdots \cup D_s \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_r$. Again, let σC denote the singular locus of C and blow up the $t = |\sigma C|$ points in σC to get $\pi_2 : X_2 \to X_1$. Let $D = \pi_2^{-1}(C) = F_1 \cup \cdots \cup F_t \cup \tilde{D}_1 \cup \cdots \cup \tilde{D}_s \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_r$ where the F_i's are the new exceptional lines. We have the following situation:

$$\begin{array}{ccccc} X_2 & \supset & D & = & F_1 \cup \cdots \cup F_t \cup \tilde{D}_1 \cup \cdots \cup \tilde{D}_s \cup \cdots \cup \tilde{B}_r \\ \downarrow & & \downarrow & & \\ X_1 & \supset & C & = & D_1 \cup \cdots \cup D_s \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_r \\ \downarrow & & \downarrow & & \\ X & \supset & B & = & B_1 \cup \cdots \cup B_r \end{array} \tag{1}$$

The divisor D forms a tri-partite graph with the following configuration

(2)



The F's are pairwise disjoint, the $\tilde{B}$'s are pairwise disjoint and the $\tilde{D}$'s are pairwise disjoint. Each F intersects exactly one of the $\tilde{B}$'s and exactly one of the $\tilde{D}$'s. Each $\tilde{D}$ intersects exactly 2 distinct F's. The F's and $\tilde{D}$'s are curves isomorphic to $\mathbb{P}^1$. The $\tilde{B}$'s consist of henselian curves and $\mathbb{P}^1$'s. We quote the following for reference.

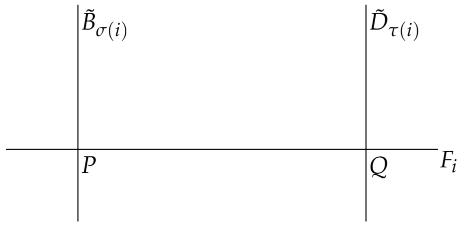
Lemma 2.2. ([A], *Lemma 1.7) Write* $D = \Gamma_1 \cup \cdots \cup \Gamma_u$ *where* Γ_i *is irreducible. Let* σD *be the singular locus of* D *and denote by* Γ_i' *the complement in* Γ_i *of those points in* σD *that lie on* Γ_i. *Set* $U = X_2 - D$. *The sequence*

$$0 \to \mathrm{Br}(U) \xrightarrow{\chi} \bigoplus_{i=1}^{u} \mathrm{H}^1(\Gamma_i', \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{\sigma D} \mathbb{Q}/\mathbb{Z} \to 0$$

is exact.

Now we show that the algebra A is essentially determined by its ramification along the divisors F_i. Denote by F_i' the complement in F_i of the singular points of D that lie on F_i. Now F_i intersects exactly one of the $\tilde{B}$'s say $\tilde{B}_{\sigma(i)}$ and one of the $\tilde{D}$'s say $\tilde{D}_{\tau(i)}$ at points P and Q as shown below.

(3)



So we see that F_i' is isomorphic to the open complement of 2 closed points in $F_i \cong \mathbb{P}^1$. We denote by $K(F_i)^h$ the henselization of the quotient field $K(F_i)$ at P. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(F_i', \mathbb{Z}/n) & \xrightarrow{\ r\ } & \mathbb{Z}/n \\ \big\downarrow & & \big\downarrow = \\ \mathrm{H}^1\big(K(F_i')^h, \mathbb{Z}/n\big) & \longrightarrow & \mathbb{Z}/n \end{array} \tag{4}$$

where r is the ramification map defined in the introduction. That r is an isomorphism follows from the Gysin sequence [M], VI, 5.4(b), (where $\mathbb{Z}/n$ is identified with $\mu_n(-1)$ via our choice of ρ),

$$0 \to \mathrm{H}^1(F_i - Q, \mathbb{Z}/n) \to \mathrm{H}^1(F_i', \mathbb{Z}/n) \xrightarrow{r} \mu_n(-1) \to \mathrm{H}^2(F_i - Q, \mathbb{Z}/n)$$

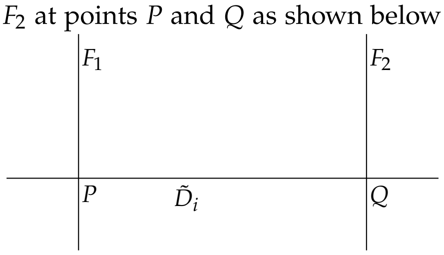
and the fact that $F_i - Q \cong \mathbb{A}^l$. The second horizontal arrow in (4) is induced from r. It is an isomorphism since $\mathrm{H}^1\big(K(F_i')^h, \mathbb{Z}/n\big) = \mathrm{Hom}\big(\mathrm{Gal}\big(K(F_i')^h\big), Z/n\big) = \mathbb{Z}/n$ by the last paragraph of the introduction.

Lemma 2.3. *Let $U = X_2 - D$. In the above context,*

$$0 \to \mathrm{Br}(U) \xrightarrow{\chi} \bigoplus_{i=1}^{t} \mathrm{H}^1(F_i', \mathbb{Q}/\mathbb{Z})$$

is exact.

Proof. Suppose the algebra A is unramified along each component of $F_1 \cup \cdots \cup F_t$. We show A is also unramified along each $\tilde{B}$ and $\tilde{D}$. This will show A is split, by Lemma 2.2. First consider one of the $\tilde{D}$'s say $\tilde{D}_i$. From the graph (2) $\tilde{D}_i$ intersects 2 F's say F_1 and F_2 at points P and Q as shown below



Say $\chi(A)$ on $\tilde{D}_i$ is the cyclic extension L. We are assuming $\chi(A)$ on each F_j is the split extension S. In Lemma 2.2 the map r sums the ramification of L at P with the ramification of the split extension S at P. Because $r\chi = 0$, we see that L is unramified at P. Likewise L is unramified at Q. So L is unramified. But $\tilde{D}_i \cong \mathbb{P}^1$ is simply connected, hence L is split. So A is unramified on $\tilde{D}_i$. Next consider a curve $\tilde{B}_i$. If $\tilde{B}_i$ is a $\mathbb{P}^1$, the above argument shows A is unramified on $\tilde{B}_i$. If $\tilde{B}_i$ is a henselian curve, then the above argument shows that L is unramified on $\tilde{B}_i$. But $\mathrm{H}^1(\tilde{B}_i, \mathbb{Q}/\mathbb{Z}) = 0$ so L is split. Thus, $\chi(A) = 0$ and A is split by Lemma 2.2. □

Combining (4) and 2.3, we define:

$$\phi : {}_n\operatorname{Br}(U) \to \bigoplus_{i=1}^{t} \mathbb{Z}/nz$$

as the composition of χ and $r : \mathrm{H}^1(F_i', \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. Here by ${}_n\operatorname{Br}(U)$ we mean the subgroup annihilated by n. Therefore ϕ is injective and associates to the algebra A a t-tuple of residues $w_1, \dots, w_t$ modulo n. The residues w_i are uniquely determined up to the conventions established in the set-up of (4), namely the choice of ρ and the choice of the point $P = F_i \cap \tilde{B}_{\sigma(i)}$ for each i.

Let $\pi_0 = \pi_2 \circ \pi_1 \circ \pi : X_2 \to \operatorname{Spec} R$ be the composite morphism. Let $E_1, \dots, E_m$. be the distinct irreducible components of the closed fiber of π_0. Each E_i is isomorphic to $\mathbb{P}^1$. Let $\mathbb{E}$ denote the additive group of divisors on X_2 generated by $E_1, \dots, E_m$. Lipman has shown [L], sections 14 and 17, that the homomorphism

$$\theta : \operatorname{Pic} X_2 \to \mathbb{E}^* = \operatorname{Hom}(\mathbb{E}, \mathbb{Z}) \tag{5}$$

given by $\theta(\Delta)(E_i) = (\Delta.E_i)(i = 1, 2, \dots, m)$ is an isomorphism since R has a rational singularity and is strictly henselian. For each E_i choose a closed point P_i such that P_i is not a singular point of D. Choose a prime divisor of Y_i on X_2 that meets E_i transversally at P_i. Then Y_i is the strict transform of a prime divisor of R. That is, each Y_i is a henselian curve on X_2 and has a unique closed point, namely P_i. So Y_i intersects $E_1 \cup \cdots \cup E_m$ exactly at the point P_i. Moreover, $(Y_i.E_j) = \delta_{ij}$ (Kronecker delta).

Lemma 2.4. *Choose $Y_1, \dots, Y_m$ as in the previous paragraph so that $(Y_i.E_j) = \delta_{ij}$. Let $X' = X_2 - Y_1 - \cdots - Y_m$. Then* $\operatorname{Pic} X' = (0)$.

Proof. We see that $\{\theta(Y_1), \dots, \theta(Y_m)\}$ generate $\mathbb{E}^*$. The homomorphism θ in (5) is an isomorphism so the Y_i must generate $\operatorname{Pic} X_2$. The result follows from [L], section 14. □

Denote by $T = T_1 \cup \cdots \cup T_u$ the intersection $D \cap X'$, where X' is as in Lemma 2.4. Since $\operatorname{Pic} X' = (0)$, the prime divisors T_i are principal. For each T_i choose a function $t_i \in K$ such that

$$\nu_\Delta(t_i) = \begin{cases} 1 & ; \Delta = T_i\,, \\ 0 & ; \text{ otherwise} \end{cases} \tag{6}$$

where Δ ranges over the prime divisors on X' and ν_Δ is the valuation on K at Δ. Re-label the functions t_i according to the notation of (1). That is, let

$$\begin{cases} b_i & \text{be the equation for } \tilde{B}_i \cap X' \\ f_i & \text{be the equation for } F_i \cap X' \\ d_i & \text{be the equation for } \tilde{D}_i \cap X' \end{cases} \tag{7}$$

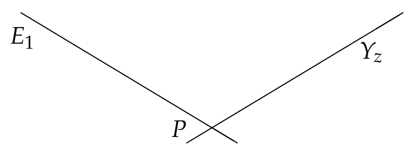
Lemma 2.5. *If $F_i \cap \tilde{B}_j = \emptyset$, then the symbol algebra $(f_i, b_j)_n$ is split.*

Proof. It suffices by [A] Lemma 1.5 to show $\chi\big((f_i, b_j)_n\big) = 0$. That is, to show that for all irreducible curves $\Delta \subseteq X_2$, $(f_i, b_i)_n$ is unramified at Δ. On symbols $(\alpha, \beta)_n$ the character map χ agrees with the tame symbol. The cyclic extension of $K(\Delta)$ afforded by $(\alpha, \beta)_n$ is obtained by adjoining $\big(\alpha^{\nu_\Delta(\beta)}\beta^{-\nu_\Delta(\alpha)}\big)^{1/n}$. Since $\nu_\Delta(f_i)$ and $\nu_\Delta(b_j)$ are zero except possibly at $\tilde{B}_j$, F_i, $Y_1, \ldots, Y_m$, the ramification divisor Γ of $A = (f_i, b_j)_n$ is contained in $\tilde{B}_j \cup F_i \cup Y_1 \cup \cdots \cup Y_m$.

Case 1: Let $\Delta = F_i$. Then $\nu_\Delta(f_i) = 1$ and $\nu_\Delta(b_j) = 0$ by (6) and (7). Since F_i is a $\mathbb{P}^1$, one of the Y's say Y_1 intersects F_i. The principal divisor (b_j) looks like $\tilde{B}_j + c_1 Y_1 + \cdots + c_m Y_m$. Thus (b_j) intersects F_i in at most one point: $Y_1 \cap F_i$. So on F_i the extension $K(F_i)(b_j^{1/n})$ ramifies at no more than one point. Such an extension is split, so A is unramified along F_i.

Case 2: $\Delta = \tilde{B}_j$ and $\tilde{B}_j$ is a $\mathbb{P}^1$. Thus $\nu_\Delta(f_i) = 0$ and $\nu_\Delta(b_j) = 1$. As in Case 1, (f_i) intersects Δ in at most one point, so A is unramified on Δ.

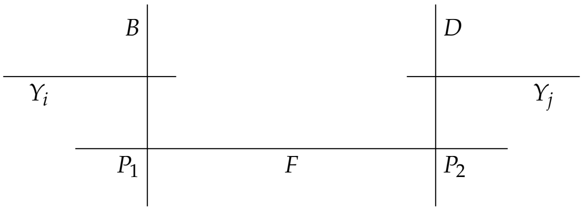
Case 3: $\Delta = Y_z$ for some z. Then Δ is a henselian curve on X_2. Since $\Gamma \subseteq \tilde{B}_j \cup F_i \cup Y_1 \cup \cdots \cup Y_m$, Cases 1 and 2 show that A is unramified along any divisor which is a $\mathbb{P}^1$. That is, A is unramified on $E_1 \cup \cdots \cup E_m$. But Y_z intersects one of the E's, say E_1.



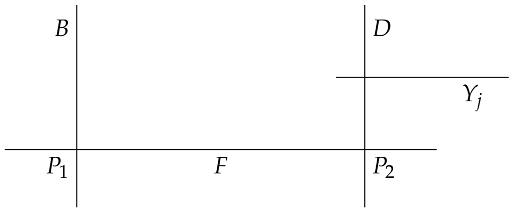
Suppose A ramifies along Y_z with Galois extension $L/K(Y_z)$. Since Y_z has just one point P, L ramifies at P. By 0.1 A must also ramify along E_1 which is a contradiction. So A is unramified along Y_z.

Case 4: We are reduced to the case $\Gamma \subseteq \tilde{B}_j$ and $\tilde{B}_j$ is a henselian curve. But $\tilde{B}_j$ intersects one of the F's say F_1. The argument of Case 3 shows that A is unramified along $\tilde{B}_j$. □

Lemma 2.6. *Let $F \in \{F_1, \ldots, F_t\}$, $B \in \{\tilde{B}_1, \ldots, \tilde{B}_r\}$, $D \in \tilde{D}_1, \ldots, \tilde{D}_s\}$. Suppose F intersects B and D at P_1 and P_2 respectively. The symbol algebra $A = (f, b/d)_n$ over K has ramification divisor*



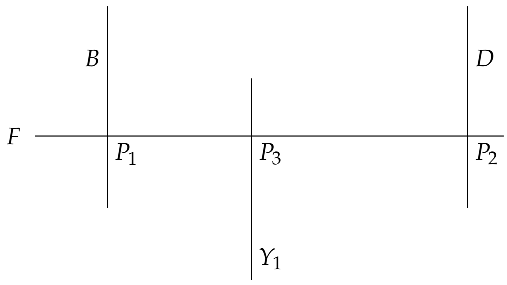
if B is an exceptional divisor (i.e., a $\mathbb{P}^1$). Otherwise B is a henselian curve and the ramification divisor is



Under the map χ the cyclic extension of $K(F)$ is obtained by adjoining the n-th root of b/d and by our definition it has ramification $+1$ at P_1.

Proof. Denote by $\bar{b}$, $\bar{d}$ the restrictions of b and d to functions on F. The extension of F' is obtained by adjoining the n-th root of $\bar{b}/\bar{d}$ because the ramification map $\mathrm{Br}(K) \xrightarrow{\chi} \mathrm{H}^1(K(F), \mathbb{Q}/\mathbb{Z})$ agrees with the tame symbol on cyclic algebras and $\nu_F(f) = 1$. Say F intersects Y_1 at P_3, so we have

(8)



The valuation of $\bar{b}$ at a closed point P of F is

$$\nu_P(\bar{b}) = \begin{cases} 1 & ; P = P_1 \\ -1 & ; P = P_3 \\ 0 & ; \text{otherwise.} \end{cases} \tag{9}$$

Indeed, the valuation at P_1 is 1 since b was chosen to be a local parameter for B at P_1. If P is not equal to P_1 or P_3, then P is not on the principal divisor (b) (on X_2). Therefore $\nu_P(\bar{b}) = 0$. Since $F \cong \mathbb{P}^1$ and $\sum \nu_P(\bar{b}) = 0$ we conclude $\nu_{P_3}(\bar{b}) = -1$. Applying a similar argument to $\bar{d}$, we see that the cyclic extension $K(F)\big((\bar{b}/\bar{d})^{1/n}\big)$ has ramification $+1$ at P_1 and -1 at P_2. Similarly A ramifies on D with cyclic extension $K(D)(f^{1/n})$. Since $D \cong \mathbb{P}^1$, A also ramifies on Y_j. If B is a $\mathbb{P}^1$, the argument is as for F and D. If B is henselian, $K(B)(f^{1/n})$ has ramification -1 at P_1. □

Proof of Theorem 2.1. Let A be a central division algebra over K of exponent n in $B(K)$ such that A is unramified on U. Suppose that on $F_1, \ldots, F_t$ the ramification data of A are $w_1, \ldots, w_t$. Consider the algebra

$$\big(f_1^{w_1} f_2^{w_2} \ldots f_t^{w_t}, b_1 \ldots b_r d_1^{-1} \ldots d_s^{-1}\big)_n \tag{10}$$

over K. Factor (10) in $\mathrm{Br}(K)$ into

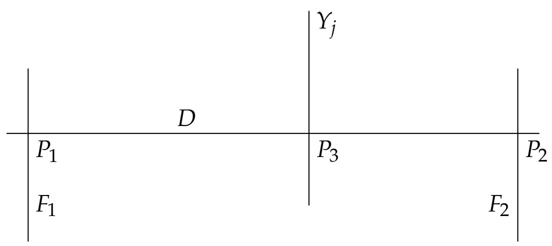
$$\prod_{i=1}^{t} \big(f_i, b_1 \ldots b_r d_1^{-1} \ldots d_s^{-1}\big)_n^{w_i} \tag{11}$$

By Lemma 2.5, (11) is Brauer-equivalent to

$$\prod_{i=1}^{t} \big(f_i, b_{\sigma(i)} d_{\tau(i)}^{-1}\big)_n^{w_i} \tag{12}$$

where F_i intersects $\tilde{B}_{\sigma(i)}$ and $\tilde{D}_{\tau(i)}$ as in (3). By Lemma 2.6 (12) has ramification data w_i on F_i. To show that (12) is unramified on U, it suffices to show (12) is unramified along each Y_j. From Lemma 2.6, it suffices to check only those Y_j that intersect $\tilde{B}$'s or $\tilde{D}$'s. Choose a D. Then D intersects two F's say F_1, and F_2 as shown below.

(13)



For a divisor Δ we denote by $\chi_\Delta(A)$ the cyclic extension of $K(\Delta)$ for A. Then $\chi_{F_1}(A)$ has ramification $-w_2$ at P_2. Thus, $\chi_D(A)$ has ramification w_1 at P_1 and w_2 at P_2. Since A is unramified along Y_j we have $w_1 + w_2 = 0$. Thus

$$(14) \qquad \left(f_1, b_{\sigma(1)} d_{\tau(1)}^{-1}\right)_n^{w_1} \left(f_2, b_{\sigma(2)} d_{\tau(2)}^{-1}\right)_n^{w_2}$$

is unramified on Y_j. Using Lemma 2.6 we conclude (12) is unramified on Y_j. Similarly we prove that (12) is unramified along the remaining Y's. So (12) and hence (10) is unramified on U. By Lemma 2.3, A is Brauer-equivalent to (10). Thus (10) has exponent n hence is a division algebra ([Re], Corollary 30.7) and is isomorphic to A. □

References

[A] M. Artin, *Two dimensional orders of finite representation type*, Manuscripta Math. **58** (1987), 445–471.

[A2] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.

[Al] A. A. Albert, *Structure of algebras*, Colloquium Publications, vol. 24, Amer. Math. Soc., Providence, RI, 1961.

[AB] M. Auslander and A. Brumer, *Brauer groups of discrete valuation rings*, Nederl. Akad. Wetensch. Proc. Ser. A **71** (1968), 286–296.

[AM] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95.

[B] J. F. Boutot, *Singularitiés rationelles et quotient par les groupes réductifs*, Invent. Math. **88** (1987), no. 1, 65–68.

[CF] J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Thompson Publ. Co., Washington, D.C., 1967.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York/Berlin, 1977.

[L] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 195–280.

[M] J. Milne, *Etale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[OS] M. Orzech and C. Small, *The Brauer group of commutative rings*, Lecture Notes in Pure and Appl. Math., vol. 11, Marcel Dekker, New York, 1975.

[R] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Math., vol. 169, Springer-Verlag, Berlin, 1970.

[Re] I. Reiner, *Maximal orders*, L.M.S. Monographs, vol. 5, Academic Press, London/New York, 1975.