ON THE BRAUER GROUP OF SURFACES AND SUBRINGS OF $k[x,y]$

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Dedicated to Goro Azumaya

In this paper we summarize some results concerning the Brauer group of classes of Azumaya algebras defined on a surface. These general results are applied to determine the Brauer group of some subrings of the polynomial ring in two variables over an algebraically closed field.

In Section 1 we let X denote a nonsingular surface of finite type over an algebraically closed field k of characteristic zero. If X is complete and the Kodaira dimension $\kappa(X) = -1$ then the Brauer group $\mathrm{B}(X)$ of X is trivial (Theorem 1.1(a)) and if X is not necessarily complete in some cases $\mathrm{B}(X)$ is determined by the dual of the algebraic fundamental group of the 'curve at infinity' on X (Theorem 1.2).

In Section 2 the problem of analyzing the Brauer group under the resolution of a normal singularity on a surface X defined over an algebraically closed field is discussed. This problem has already been studied by several authors including [3], [7], [12], [15], [17], [25], and [26]. In [7] a summary of what was known until that time was given. Theorem 2.8 summarizes our knowledge on the resolution problem for both B(X) and the cohomological Brauer group $\mathrm{B}'(X)$. If k is the field of complex numbers then connections with algebraic topology can be made (Theorem 2.9) which permit us to explain some of

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the phenomena given in [7]. Finally, we point out that if R is a normal two-dimensional graded k-algebra of finite type with characteristic of $k = 0$ and $R_0 = k$ and if the only singularities on $\operatorname{Spec} R$ are rational then $\mathrm{B}(R) = 0$.

In Section 3 the general results are applied to some examples. In particular, if k is an algebraically closed field of characteristic zero and R is a normal affine subring of $k[x,y]$ with $k[x,y]$ finitely generated as an R-module, then $\mathrm{B}(R) = 0$ whenever R is graded, or $k = \mathbb{C}$. Two examples are given. The Brauer group of the cone over a smooth integral curve defined over an algebraically closed field of characteristic zero is trivial. Using a Mayer-Vietoris sequence an example of a non-normal affine subring R of $k[x,y]$ with $k[x,y]$ a finite R-algebra and $\mathrm{B}(R) \neq 0$ is given. This shows the hypothesis that the ring R be normal in Theorem 3.1 is necessary.

Our techniques and basic terminology follow A. Grothendieck [17]. In particular, unless otherwise specified, all cohomology is in the étale topology. By a surface we mean an integral, separated, locally Noetherian two-dimensional scheme over the algebraically closed field k.

1

In this section let X denote a nonsingular surface of finite type over an algebraically closed field k of characteristic zero. If X is complete, following [18] the Kodaira dimension $\kappa(X)$ is defined to be the transcendence degree over k of the ring $R = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathscr{L}(nK))$ minus 1, where K is the canonical divisor on X and cohomology is in the Zariski topology. For a surface X, $-1 < \kappa(X) < 2$. Theorem 6.1 of [18] asserts $\kappa(X) = -1$ if and only if X contains an affine open subset U isomorphic to $\mathbb{A}^1 \times \Gamma$ where Γ is a nonsingular curve. Let D denote the regular completion of Γ.

Theorem 1.1. [12] *Let X be a complete nonsingular surface of finite type over k with the Kodaira dimension* $\kappa(X) = -1$. *If D is as above, then*

a) $\mathrm{H}^2(X, \mathbb{G}_m) = 0$

b) $\mathrm{H}^3(X, \mathbb{G}_m) = \mathrm{Hom}(\pi_1(D), \mathbb{Q}/\mathbb{Z})$,

c) X is rational if and only if $\mathrm{H}^3(X, \mathbb{G}_m) = 0$.

The proof of Theorem 1.1 can be found in [12]. M. Artin pointed out to us examples of complete nonsingular surfaces X over k with $\mathrm{B}(X) \neq 0$. The calculation in (b) of Theorem 1.1 is important in the analysis of noncomplete surfaces.

If X is not necessarily complete then X can be embedded as an open subset of a complete nonsingular surface S. Let $Z = S - X$ and write $Z = Z_1 \cup \cdots \cup Z_m$ where the Z_i are the connected components of Z. If $\kappa(S) = -1$, then S contains an affine open subset isomorphic to $\mathbb{A} \times \Gamma$ where Γ is a nonsingular curve. Let D be the nonsingular completion of Γ.

Theorem 1.2. [12] *Let X be a nonsingular surface of finite type over k. Embed X as an open subset of a complete nonsingular surface S as above. Assume $\kappa(S) = -1$ and let Z_i, D be as above. If the pair (S,Z) satisfies the "Theorem of Purity for the Brauer group"* [17, 6.2, III], *then*

a)

$$0 \to \mathrm{B}(X) \to \bigoplus_i \mathrm{Hom}(\pi_1(Z_i), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\pi_1(D), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^3(X, \mathbb{G}_m)$$

is exact.

b) If X is affine, then

$$0 \to \mathrm{B}(X) \to \bigoplus_i \mathrm{Hom}(\pi_1(Z_i), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\pi_1(D), \mathbb{Q}/\mathbb{Z}) \to 0$$

is exact.

c) If X is rational, then

$$\mathrm{B}(X) \cong \bigoplus_i \mathrm{Hom}(\pi_1(Z_i), \mathbb{Q}/\mathbb{Z})$$

Theorem 1.2 is proved in [12]. If $k = \mathbb{C}$ in Theorem 1.2 then Z_i, D can be viewed as real 2-manifolds. The algebraic fundamental group $\pi_1(Z_i)$ and the topological fundamental group have the same finite quotients [21, p. 40]. It is well known that the topological fundamental group of a compact connected real 2-manifold M of genus g is the direct sum of g-copies of $\mathbb{Z}$ together with one copy of $\mathbb{Z}/(2)$ if M is not orientable. In particular, if X is as in Theorem 1.2 and the Z_i are simply connected then $\mathrm{B}(X) = 0$. If Z is regular the pair (S,Z) always satisfies the conclusion of the "Theorem of Purity for the Brauer group".

2

In this section let X denote a normal, locally Noetherian, integral, separated, two-dimensional scheme over an algebraically closed field k. Let K denote the function field of X and $\mathrm{B}'(X)$ the cohomological Brauer group $\mathrm{H}^2(X, \mathbb{G}_m)$. For a point p on X let

$\mathscr{O}_p$ denote the local ring of X at p, $\mathscr{O}_p^h$ the Henselization of $\mathscr{O}_p$, and $\hat{\mathscr{O}}_p^h$ the completion of $\mathscr{O}_p$. The surface X has at most finitely many singular points. The next two results due to A. Grothendieck and B. Auslander, respectively, show that to study the map $\mathrm{B}'(X) \to \mathrm{B}(K)$, it suffices to study the corresponding maps on the local rings at the singular points of X.

Theorem 2.1. [17, II, sec. 1] *If X is a surface with singular points $p_1, \dots, p_n$, then the sequence*

$$0 \to \bigoplus_i \mathrm{Cl}\left(\mathscr{O}_{p_i}^h\right) / \mathrm{Cl}\left(\mathscr{O}_{p_i}\right) \xrightarrow{\phi} \mathrm{B}'(X) \to \mathrm{B}(K)$$

is exact where $\mathrm{Cl}\left(\mathscr{O}_{p_i}\right)$ is the divisor class group of $\mathscr{O}_{p_i}$. Moreover, ϕ is an isomorphism modulo torsion subgroups.

Theorem 2.2. [3] *If X is an affine surface then the sequence*

$$0 \to \mathrm{B}(X) \to \prod_p \mathrm{B}\left(\mathscr{O}_p\right)$$

is exact where the product runs over all points p on X.

Corollary 2.3. *If X is an affine surface, the sequence*

$$0 \to \mathrm{B}(X) \to \left(\bigoplus_{p \in \mathrm{Sing}(X)} \mathrm{B}\left(\mathscr{O}_p\right) \right) \oplus \mathrm{B}(K)$$

is exact.

Let Ω be a singular point of X and $\pi : Y \to X$ a series of blowings-up over Ω. Let $Y^h = Y \times_X \mathrm{Spec}\left(\mathscr{O}_\Omega^h\right)$. The following theorem was proved by W. Gordon using the Leray spectral sequence

$$\mathrm{H}^p\left(X, R^q\pi_*\mathbb{G}_m\right) \implies \mathrm{H}^{p+q}\left(Y, \mathbb{G}_m\right) .$$

Theorem 2.4. [15] *With X, Y as above, there is an exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(Y) \to \mathrm{Pic}\left(Y^h\right) \to \mathrm{B}'(X) \to \mathrm{B}'(Y) \to 0 .$$

Let R be a two-dimensional normal local ring with maximal ideal M and algebraically closed residue field k. Let $f : X \to \mathrm{Spec}(R)$ be a desingularization of R. Let $E_1, \dots, E_n$ be the irreducible components of the closed fiber E, i.e., all the integral curves on X with exceptional support. Then $f^{-1}\left(\{M\}\right)_{red} = E_1 + \cdots + E_n$. It is known that the intersection matrix $\left((E_i \cdot E_j)\right)$ is negative definite [20, Lemma 14.1]. Let $\mathbb{E}$ be the additive group of divisors on X with exceptional support, i.e., divisors of the form $\sum s_i E_i$. For each i let $d_i > 0$ be the greatest common divisor of all the degrees of invertible sheaves on E_i. Define $\theta : \mathrm{Pic}(X) \to \mathbb{E}^* = \mathrm{Hom}(\mathbb{E}, \mathbb{Z})$ by $\left(\theta(\Delta)\right)(E_i) = \dfrac{1}{d_i}\left(\Delta \cdot E_i\right)$.

Denote by $\operatorname{Pic}^0 X$ the kernel of θ and by G the cokernel of θ. Let $U \cong X - f^{-1}(\{M\}) = \operatorname{Spec} R - \{M\}$. The following diagram with exact rows and columns is due to J. Lipman [20].

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{E} \cap \operatorname{Pic}^0 X = 0 & \longrightarrow & \mathbb{E} & \longrightarrow & \theta(\mathbb{E}) & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} X & \xrightarrow{\theta} & \mathbb{E}^* & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \rho \operatorname{Pic}^0 X & \longrightarrow & \operatorname{Pic} U & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

The group H is defined by the diagram. Since $U = \operatorname{Spec}(R) - \{M\}$ is regular, $\operatorname{Pic}(U) = \operatorname{Cl}(U) = \operatorname{Cl}(R)$. The singularity of R is said to be a rational singularity if $\operatorname{Cl}(R)$ is finite. From [20], one of R, R^h, $\hat{R}$ has a rational singularity if and only if they all do. Using [20, Prop. 16.3] with $A = R$ and $B = R^h$ and [20, Prop. 17.1] one can show that $H = \operatorname{Cl}\left(R^h\right)$ if the singularity of R is rational.

Theorem 2.5. *Let R be a two-dimensional normal local ring with an algebraically closed residue field. If R has a rational singularity then* $\mathrm{B}(R) = \mathrm{B}'(R)$ *and the following diagram commutes and has exact rows and columns*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{E} & \longrightarrow & \theta(\mathbb{E}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Pic} X & \xrightarrow{\theta} & \mathbb{E}^* & \longrightarrow & \mathrm{B}(R) & \longrightarrow & \mathrm{B}(K) \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
0 & \longrightarrow & \operatorname{Cl}(R) & \longrightarrow & \operatorname{Cl}\left(R^h\right) & \longrightarrow & \mathrm{B}(R) & \longrightarrow & \mathrm{B}(K) \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Proof. For exactness of the diagram apply to the diagram of J. Lipman, the preceding comments and Theorem 2.1. To see that $\mathrm{B}(R) = \mathrm{B}'(R)$ note that $\operatorname{Cl}\left(R^h\right)$ is finite so by Theorem 2.1 $\mathrm{B}'(R)$ is torsion. But for an affine scheme X, $\mathrm{B}(X)$ is the torsion subgroup of $\mathrm{B}'(X)$ by a theorem of O. Gabber [14]. Thus $\mathrm{B}'(R) = \mathrm{B}(R)$. □

Corollary 2.6. *If X is a normal, integral, locally noetherian, two-dimensional scheme of finite type over an algebraically closed field k with only rational singularities, then* $\mathrm{B}'(X)$ *is torsion.*

Proof. Apply Theorem 2.1 and Theorem 2.5. □

Corollary 2.7. *With the hypothesis of Theorem 2.5, let* $E = f^{-1}(\{M\})$ *and* $E^h = E \times_{\operatorname{Spec} R} \operatorname{Spec} R^h$. *Then* $\operatorname{Pic} E^h = \operatorname{Pic} E$.

Proof. Let X be a desingularization of $\operatorname{Spec} R$. From Theorem 2.4 and Theorem 2.5 we have two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Pic} E^h & \longrightarrow & \mathrm{B}'(R) & \longrightarrow & \mathrm{B}(K) \\ & & \downarrow = & & \downarrow \theta & & \downarrow = & & \downarrow = \\ 0 & \longrightarrow & \operatorname{Pic} X & \longrightarrow & \mathbb{E}^* & \longrightarrow & \mathrm{B}(R) & \longrightarrow & \mathrm{B}(K) \end{array}$$

Both $\mathbb{E}^*$ and $\operatorname{Pic} E$ are free on n generators. From [20, Lemma 14.3] the restriction map ϕ : $\operatorname{Pic} E^h \to \operatorname{Pic} E$ is surjective. Since $\operatorname{Pic} E \cong \mathbb{Z}^{(n)}$, the map ϕ splits. The diagram commutes so ϕ is an isomorphism. □

Let R denote a local normal domain, let $\{(S_i, m_i)\}_{i \in I}$ be a directed family of Galois coverings with fixed maximal ideals. The derived family of local rings $\left\{(S_i)_{m_i}\right\}_{i \in I}$ is directed. Let $\tilde{R} = \varinjlim (S_i)_{m_i}$ and let $\pi_1 = \varprojlim \operatorname{Gal}(S_i/R)$. Now we summarize the preceding results. By using Theorem 2.1, the following can be extended to any finite number of singular points. Denote by $\mathrm{B}'(K/X)$ the kernel of $\mathrm{B}'(X) \to \mathrm{B}(K)$.

Theorem 2.8. *Let X be a normal, integral, locally noetherian, two-dimensional scheme over the algebraically closed field k. Assume p is the only singular point on X, then*

a) $\mathrm{B}'(K/X) \cong \operatorname{Pic}(Y^h)/\mathscr{P}$ *where* $\mathscr{P} = \operatorname{Pic} Y / \operatorname{Pic} X$, *and Y nonsingular is obtained from X by a series of blowings-up over p.*

b) $\mathrm{B}'(K/X) \cong \mathrm{B}'(K/\mathscr{O}_p)$.

c) $\mathrm{B}'(K/X) \cong \operatorname{Cl}(\mathscr{O}_p^h) / \operatorname{Cl}(\mathscr{O}_p)$.

d) $\mathrm{B}'(K/X) \cong \operatorname{Cl}(\hat{\mathscr{O}}_p) / \operatorname{Cl}(\mathscr{O}_p)$.

e) $\mathrm{B}'(K/X) \cong \operatorname{Cl}(\tilde{\mathscr{O}}_p)^{\pi_1} / \operatorname{Cl}(\mathscr{O}_p)$.

Moreover, if the singularity on X is rational,then

f) $\mathrm{B}(K/X) \cong \operatorname{Pic}(E)/\mathscr{P}$.

g) $\mathrm{B}(K/X) \cong \mathrm{B}(K/\mathscr{O}_p)$.

h) $\mathrm{B}(K/X) \cong \operatorname{Cl}(\mathscr{O}_p^h) / \operatorname{Cl}(\mathscr{O}_p)$.

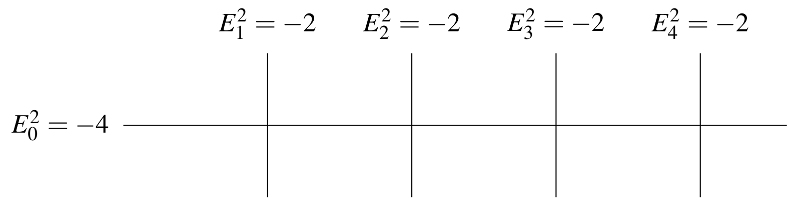
i) $\mathrm{B}(K/X) \cong \operatorname{Cl}(\hat{\mathscr{O}}_p) / \operatorname{Cl}(\mathscr{O}_p)$.

The proof of e) is in [25]. In the context of Theorem 2.8, $\mathrm{Cl}(\hat{\mathscr{O}}_p) \cong \mathrm{Cl}(\mathscr{O}_p^h)$. In [15] W. Gordon shows that if X has a rational singularity and $\operatorname{char} k > 0$, then $\mathrm{B}(K/X) \neq 0$. It follows from results of Hoobler published in these proceedings that $\mathrm{B}'(K/X)_{tor} = \mathrm{B}(K/X)$ in a) – d) above.

Let X be an algebraic surface over $\mathbb{C}$. Following [5] a singular point x on X is said to be given by a group action if there is an isomorphism from the local ring $\mathscr{O}_{X,x}$ of X at x into $\mathscr{O}_{\mathbb{A}^2,p}$ (where p is the origin $(0,0)$ of $\mathbb{A}^2$) and a finite group G of automorphisms of $\mathscr{O}_{\mathbb{A}^2,p}$ so that $\mathscr{O}_{\mathbb{A}^2,p}^G = \mathscr{O}_{X,x}$. It is shown in [5], [23] that a singular point x on a surface X is given by a group action if and only if $\pi_{X,x}$ is finite where $\pi_{X,x}$ is the topological fundamental group of $N - \{x\}$ where N is a star-like open subset of X (in the analytic topology) containing x.

Theorem 2.9. [12] *Let X be a normal complex affine algebraic surface and assume $\pi_{X,x}$ is finite for each singular point x on X. Then* $\mathrm{B}(K/X) = 0$.

In [22] D. Mumford calculated the fundamental group $\pi_{X,x}$ of an isolated singular point x on a complex surface X in terms of generators and relations determined by the geometry of the exceptional line on a resolution of the singularity x. In particular, if x is resolved by a single blow-up and the exceptional curve E has self intersection number $-l$, then $\pi_{X,x}$ is a cyclic group of order l. Theorem 2.9 implies that $\mathrm{B}(K/X) = 0$ when x is the only singularity on X. This answers a question raised in [7]. If the exceptional curve E has irreducible components $E_0, E_1, \ldots, E_4$ with configuration



and self-intersections as given, then using [22] one can show $\pi_{X,x}$ is infinite. Hence the singularity at x is not given by a group action.

Theorem 2.10. [12] *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a normal, graded, two-dimensional k-algebra of finite type with $R_0 = k$. Assume k is algebraically closed and* $\operatorname{char} k = 0$. *If the only singularities on* $\operatorname{Spec} R$ *are rational, then* $\mathrm{B}(R) = 0$.

A normal domain R is said to have a discrete divisor class group (DCG) in case $\mathrm{Cl}(R) = \mathrm{Cl}\big(R[[x]]\big)$. The proof of Theorem 2.10 uses the fact that $\mathrm{B}(R) \cong \mathrm{B}\big(R[x]\big)$, which is a consequence of the next theorem due to P. Griffith.

Theorem 2.11. [16] *Let R be a normal domain containing a field of characteristic zero. If the strict Henselization of R at each prime ideal has DCG, then there is a natural isomorphism* $\mathrm{B}(R) \cong \mathrm{B}\big(R[x]\big)$.

Note that Theorem 2.10 is valid with the hypothesis "Spec R has only rational singularities" replaced by "R has DCG". From [9] and [10] it follows that R has DCG when $\mathrm{Cl}\left(R_p^h\right)$ is finitely generated for each maximal ideal p.

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Theorem 3.1. [12] *Let k be an algebraically closed field of characteristic zero and let R be an affine normal subring of $k[x,y]$ such that $k[x,y]$ is a finitely generated R-module. Then* $\mathrm{B}(R) = 0$ *whenever R is graded or $k = 0$.*

Note that with the hypothesis of Theorem 3.1 there is a finite surjective morphism from $\mathbb{A}^2$ to $X = \operatorname{Spec} R$. Thus each singularity on X is a quotient singularity and is given by a group action. In particular, each singular point on X is rational, and the first conclusion of the theorem follows from 2.10. If $k = \mathbb{C}$, then the topological fundamental group at each point of $\operatorname{Spec} R$ is finite so by 2.9 $\mathrm{B}(K/R) = 0$. By utilizing results in [1] and [29] it can be shown that $\mathrm{B}(X) = \mathrm{B}(X - Y) = 0$ where Y is the set of singular points on X.

Theorem 3.2. [12] *Let k be an algebraically closed field of characteristic zero, and let $F \in k[x,y,z]$ be a homogeneous polynomial. If $R = k[x,y,z]/(F)$ and* $\operatorname{Proj} R = Y \subseteq \mathbb{P}^2$ *is a smooth integral curve, then* $\mathrm{B}(R) = 0$.

Proof. R is the affine coordinate ring of the cone $X = \mathrm{Spec}(R)$ over Y in $\mathbb{A}^3$. The ring R is a two-dimensional normal graded ring and has one singular point at (0,0,0). In general, the singularity of X is not rational so Theorem 2.10 does not apply. It follows from [8] that $\mathrm{Cl}(\hat{R}_p) \cong \mathrm{Cl}(R_p) \oplus V$ where V is a finite dimensional vector space over k. Therefore, $\mathrm{Cl}(\hat{R}_p)/\mathrm{Cl}(R_p)$ is torsion free.

From Theorem 2.1 we have $\mathrm{B}'(K/R)$ is torsion free. Since $\mathrm{B}(R)$ is torsion we have $\mathrm{B}(K/R) = 0$. Let $\phi : \tilde{X} \to X$ be the morphism obtained by blowing up (0,0,0) in $\mathbb{A}^3$. It is an exercise [18, I, 5.7] to show X is regular and $\phi^{-1}(0,0,0) \cong Y$. It follows from Theorem 1.2 that $\mathrm{B}(\tilde{X}) = 0$. Since $\tilde{X}$ is regular we have an exact commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{B}(\tilde{X}) & \longrightarrow & \mathrm{B}(K) \\ & & \big\uparrow & & \big\uparrow = \\ 0 & \longrightarrow & \mathrm{B}(X) & \longrightarrow & \mathrm{B}(K) \end{array}$$

Therefore $\mathrm{B}(X) \subseteq \mathrm{B}(\tilde{X}) = 0$ So $\mathrm{B}(X) = \mathrm{B}(R) = 0$. □

Finally, we give an example of an affine subring R of $k[x,y]$ with $k[x,y]$ a finite R-algebra and k algebraically closed of characteristic zero yet $\mathrm{B}(R) \neq 0$. This shows the condition that R be normal in Theorem 3.1 is necessary. To begin let R be any domain with integral closure $\bar{R}$ and conductor ideal c. Assume $\bar{R}$ is a finite R-algebra, then the diagram

$$\begin{array}{ccc} R & \longrightarrow & \bar{R} \\ \big\downarrow & & \big\downarrow \\ R/c & \longrightarrow & \bar{R}/c \end{array}$$

is a cartesian square with respect to the units functor in the sense of 1.3 of [13]. By Proposition 1.3(b) of [13] the sequence

$$\cdots \to \mathrm{H}^n(R,U) \to \mathrm{H}^n(\bar{R},U) \oplus \mathrm{H}^n(R/c,U) \to \mathrm{H}^n(\bar{R}/c,U) \to \mathrm{H}^{n+1}(R,U) \to \dots$$

is exact. Taking torsion subgroups and identifying the terms of low degree we obtain the Mayer-Vietoris sequence [6] and [19]

$$\begin{aligned} 1 \to U(R)_t \to U(\bar{R})_t \oplus U(R/c)_t \to U(\bar{R}/c)_t \to \\ \mathrm{Pic}(R)_t \to \mathrm{Pic}(\bar{R})_t \oplus \mathrm{Pic}(R/c)_t \to \mathrm{Pic}(\bar{R}/c)_t \to \\ \mathrm{B}(R) \to \mathrm{B}(\bar{R}) \oplus \mathrm{B}(R/c) \to \mathrm{B}(\bar{R}/c) \end{aligned}$$

Let $R = k[x, y^2, y(y^2 - p(x))]$ where $p(x)$ is some polynomial in x. The quotient field of R is $k(x,y)$, the integral closure of R is $k[x,y]$, the conductor c viewed as an ideal in $k[x,y]$ is

$(y^2 - p(x))$ and $c = ((y^2 - p(x), y(y^2 - p(x)))$ when viewed as an ideal in R. Also $k[x,y]$ is generated by the elements 1, y as an R-module. Moreover $\bar{R}/c = k[x,y]/(y^2 - p(x))$ and $R/c = k[x,z]/(z - p(x)) \cong k[x]$ where z corresponds to y^2.

If k is an algebraically closed field of characteristic 0, then $\mathrm{Pic}(R/c) = \mathrm{Pic}(\bar{R}) = \mathrm{B}(\bar{R}) = \mathrm{B}(R/c) = 0$ so from the Mayer-Vietoris sequence $\mathrm{Pic}(\bar{R}/c)_t = \mathrm{B}(R)$. If we let $p(x) = x^2(x+1)$, then R/c is the coordinate ring of the nodal cubic and by applying the Mayer-Vietoris sequence to R/c we have $\mathrm{Pic}(R/c)_t \cong U(k)_t$ so $\mathrm{B}(R) \cong U(k)_t \neq 0$ in this case.

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