Investigating Algebraic Surfaces Associated to Elliptic Curves

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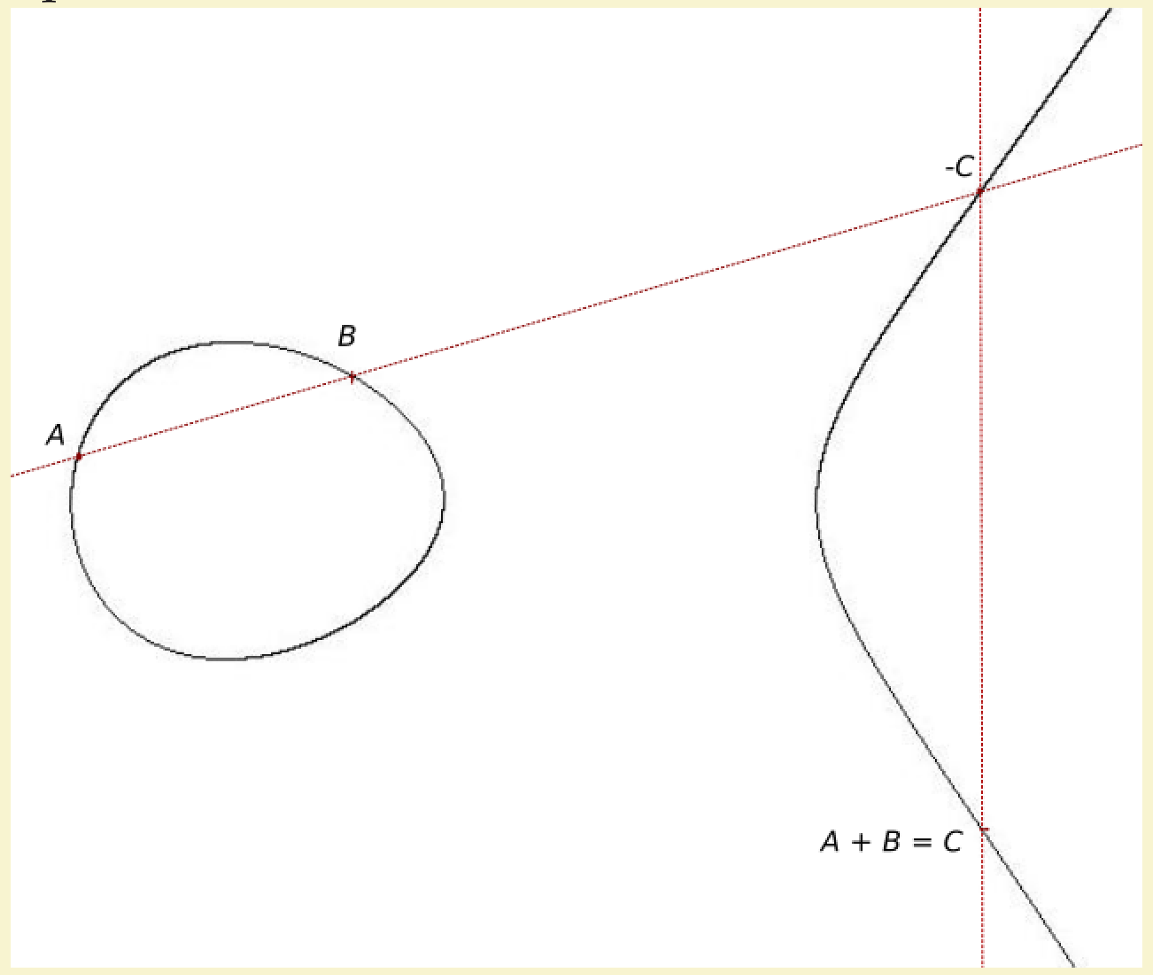
Group Law on an Elliptic Curve

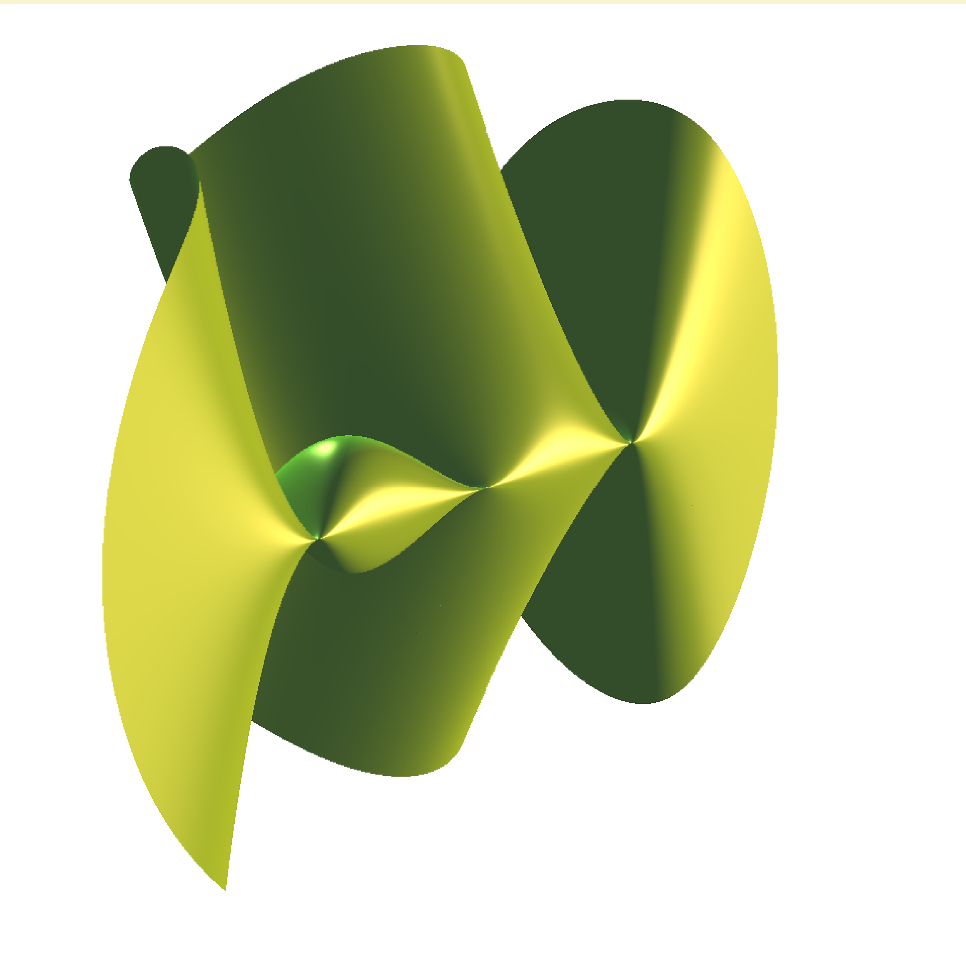
There is a natural group law on the points of an elliptic curve.



For example, many cryptosystems in use today are based on these groups.

A Family of Nonnormal Surfaces

The surface $z^2 = y(y - x(x^2 - 2))^2$



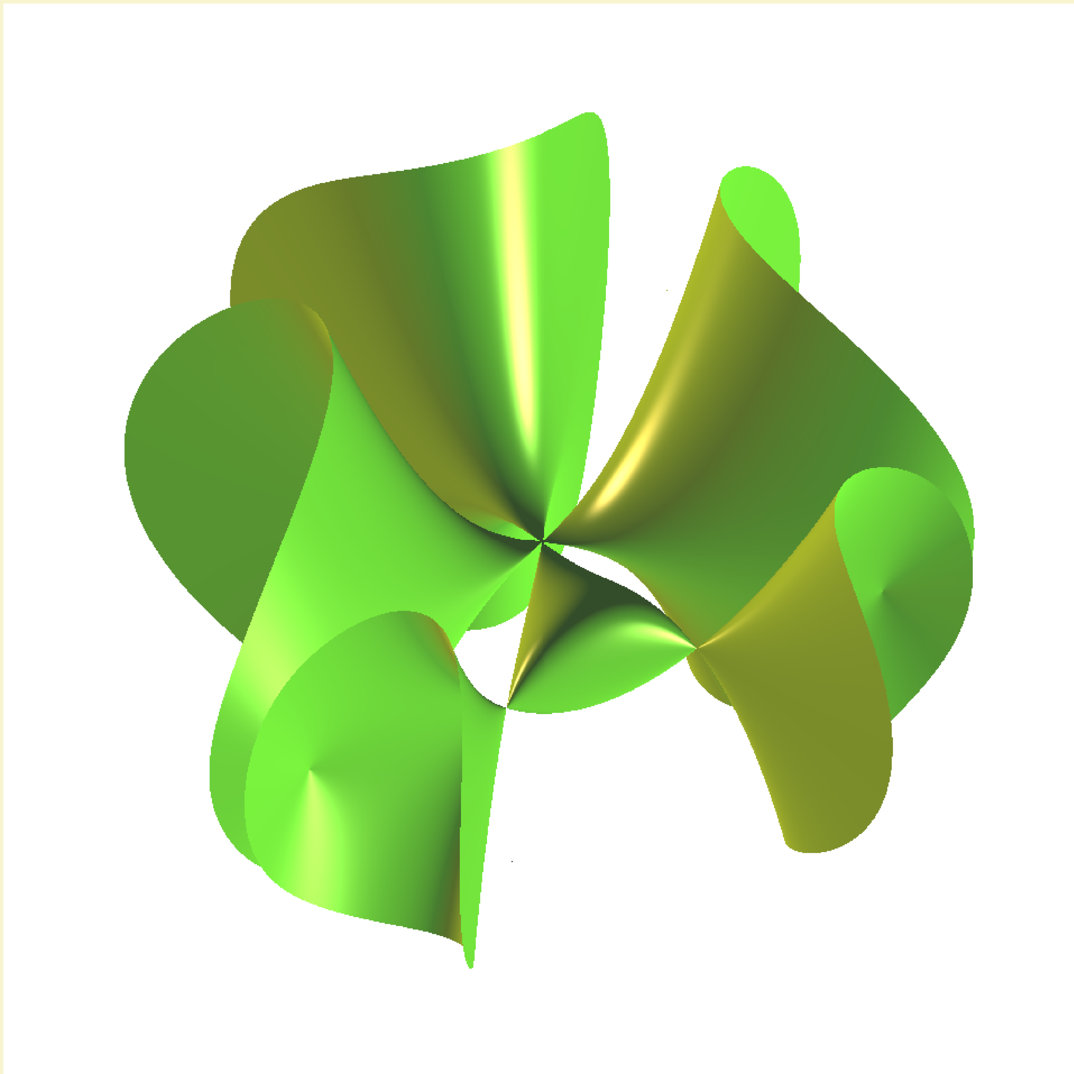
is nonnormal. The normalization has an elliptic curve isomorphic to the plane cubic: $y^2 = x(x^2 - 2)$. Ref: [(with F. R. DeMeyer) *On the Brauer group of surfaces and subrings of* $k[x, y]$, Brauer Groups in Ring Theory and Algebraic Geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin-New York, 1982, pp. 211–221.] and [*A Family of Nonnormal Double Planes Associated to Hyperelliptic Curves*, preprint.]

From Elliptic Curves to Algebras on Surfaces

The Invariant: To study algebraic surfaces, critical invariants are defined. This investigation focuses on the invariant called the Brauer group.
The Importance: The Brauer group parametrizes the Azumaya algebras on the surface. The importance of the Brauer group invariant is due to its subtle connections to the algebraic, arithmetic, and topological properties of the surface.
Computations: The family of varieties for which the Brauer group invariant has been computed is quite small. In the last fifty years much progress has been made on the problem of getting a better understanding of this invariant, especially for algebraic varieties of low dimension. Curves and surfaces receive a lot of attention because in many cases the computations can be completely carried out.
The Examples: For each surface shown here there is an associated elliptic curve. Nontrivial Azumaya algebras are constructed on the surface using the group law on the associated elliptic curve. These algebras represent nontrivial elements in the Brauer group of the surface, hence demonstrating the existence of important algebraic, arithmetic, and topological features.
Reference: [*Separable algebras*, Graduate Studies in Mathematics, vol. 183, American Mathematical Society, Providence, RI, 2017.].

A Nonrational Singularity

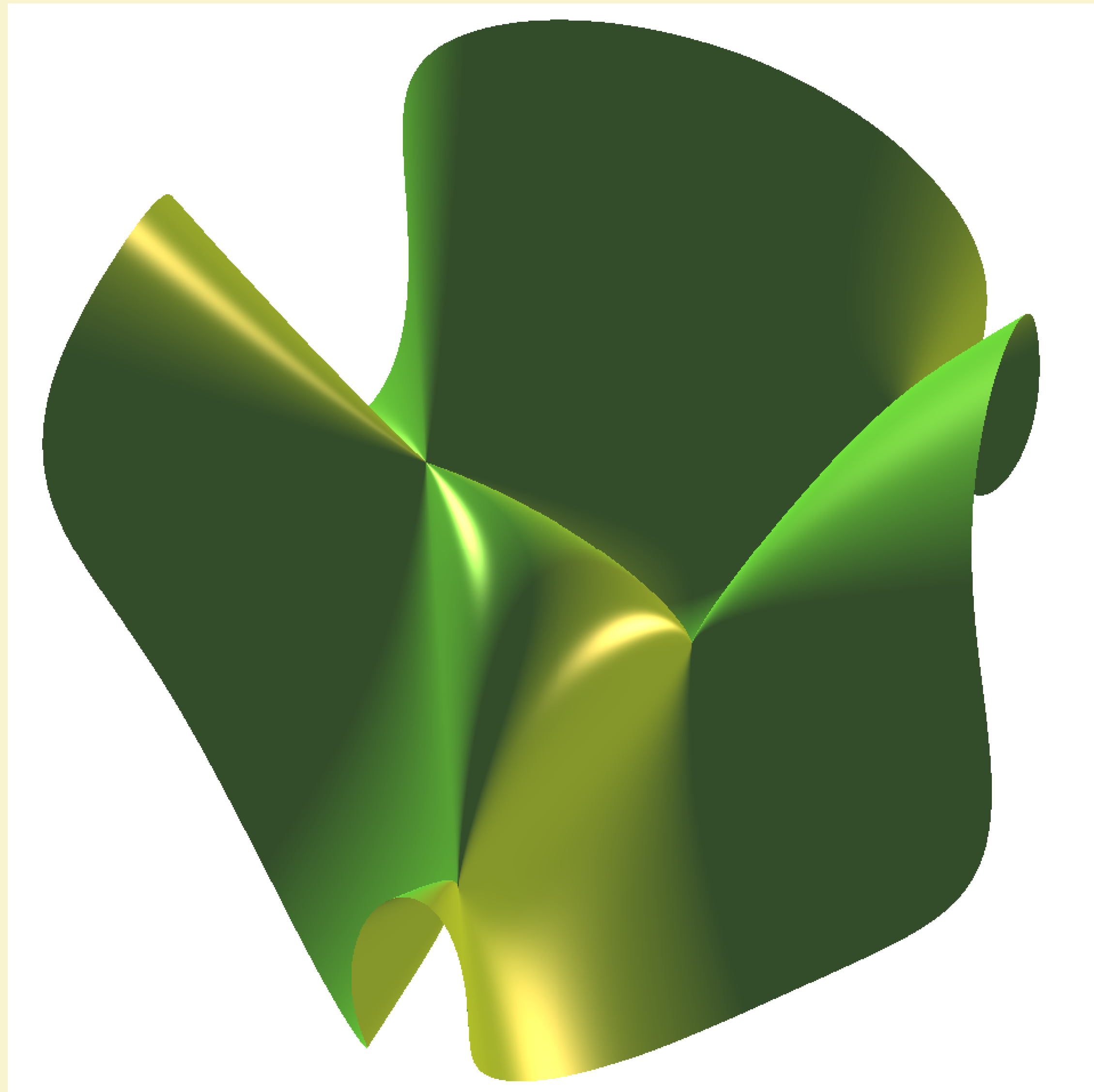
The surface $z^2 - (y^2 - 3x^2)(3y^2 - x^2)(1 - x) = 0$



has a nonrational singularity at the origin. A resolution of this singularity contains an exceptional curve that is birational to the plane quartic elliptic curve: $y^2 + (x^2 - 3)(3y^2 - 1) = 0$. Ref: [(with D. N. Bulj and D. M. Harmon) *Generically trivial Azumaya algebras on a rational surface with a non rational singularity*, Comm. Algebra **41** (2013), no. 11, 4333–4338.].

Another Nonrational Singularity

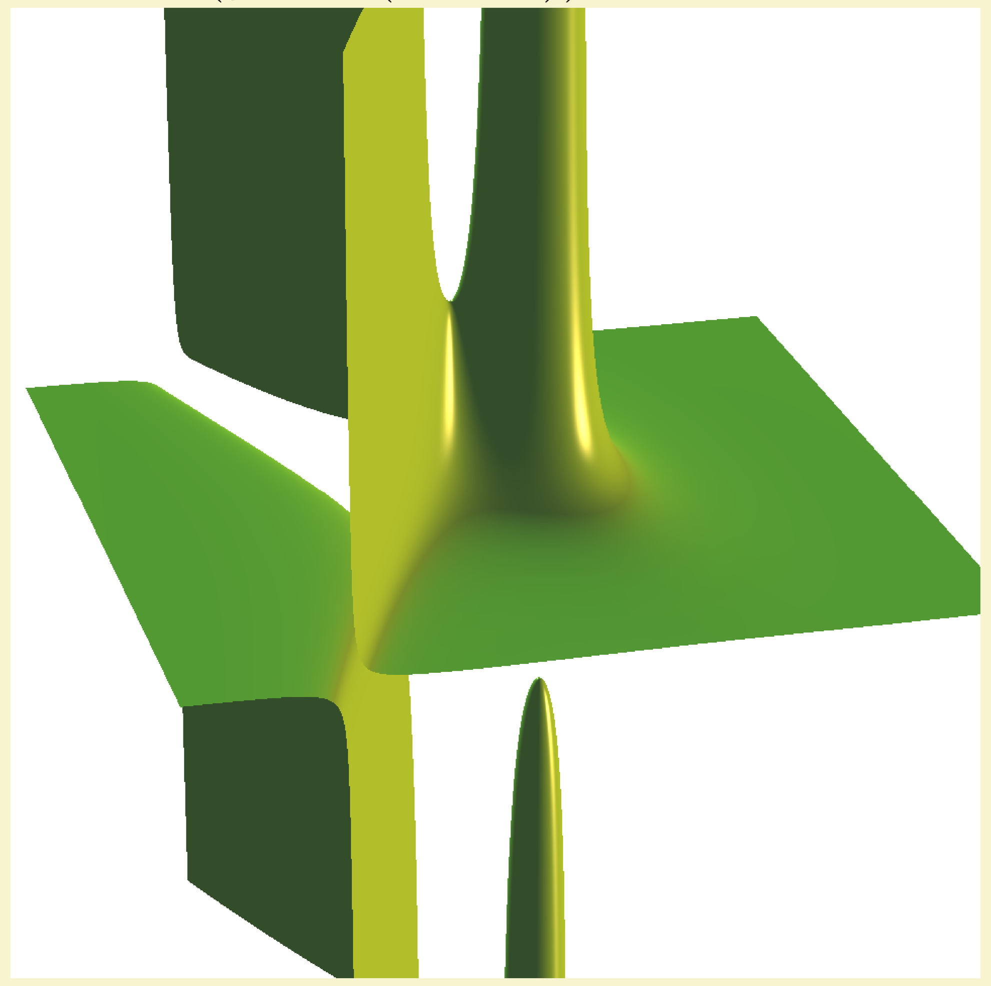
The surface $z^3 + y(y^2 - x^2)(x - 1) = 0$



has a nonrational singularity at the origin. A resolution of this singularity contains an exceptional curve that is birational to the plane cubic elliptic curve: $y^3 + x(x^2 - 1) = 0$. Ref: [(with D. M. Harmon) *The Brauer group of an affine rational surface with a non-rational singularity*, J. Algebra **388** (2013), 107–140.].

The Open Complement of a Curve

The surface $z(y^2 - x(x^2 - 1)) = 1$



corresponds to the open complement of the plane cubic elliptic curve: $y^2 = x(x^2 - 1))$. Ref: [*On the Brauer group of* $k[x_1, \cdots, x_n, 1/f]$, J. Algebra **122** (1989), no. 2, 410–424.] and [*On the Brauer group of a localization*, J. Algebra **147** (1992), no. 2, 365–378.]

An Elliptic Ruled Surface

The surface $z^2 = (y^2 - 2x^2)(y^2 - x^2))$

is locally a ruled surface and has general cross-sections that are elliptic curves. Ref: [*The Brauer group of an affine double plane associated to a hyperelliptic curve*, Comm. Algebra **45** (2017), no. 4, 1416–1442.]